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THE EXPONENTIAL OF ITERATION OF $e^x - 1$

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ABSTRACT. We study the mapping properties of a nonconstant entire solution of the equation

$$f(z+1) = e^{f(z)} - 1.$$

1. INTRODUCTION

Give a function φ on a set X to itself, we consider solutions of the Abel equations

$$(A_1) \qquad g(\varphi(z)) = g(z) + 1,$$

$$(A_2) \qquad f(w+1) = \varphi(f(w)),$$

where $f = g^{-1}$ and $w = g(z)$.

These solutions are important in studying the flow on X determined by φ, since the family $(\varphi_t)_{t\in\mathbf{R}}$ of mappings given by $\varphi_t(z) = f(g(z)+t)$ satisfies the formal identities

$$\varphi_0(z) = z, \qquad \varphi_1(z) = \varphi(z), \quad \text{and} \quad \varphi_t(\varphi_u(z)) = \varphi_{t+u}(z).$$

A solution of (A_1) was called by Szekeres [6] a logarithm of iteration of φ. Entire nonconstant solutions of (A_2) were constructed in [7] for a wide class of entire functions φ, including the special case $\varphi(z) = e^z - 1$: such functions will analogously be called exponentials of iteration.

In this paper we study some of the mapping properties of this solution of (A_2) when $\varphi(z) = e^z - 1$. We locate its zeros and asymptotic values and show that its derivative is never zero. Evidently its rate of increase exceeds any finite composition of exponential functions. We also show that it has the property A of Edrei and Erdös [3]: for some $A > 0$ the set $\{z : |f(z)| > A\}$ has finite area.

There is substantial literature concerning the asymptotic behaviour of sequences defined by iteration of analytic functions: This goes back at least to

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Fatou and Julia and is continuing vigourously at present. Our results are complementary to these since they relate to the continuous interpolation of iterates rather than their long term behaviour.

2. Construction of solutions

We summarize the results of [7] for use in later sections.
It is known from Fatou [4, pp. 191–202], that if τ is analytic in a neighborhood of 0, $\tau(0) = 0$, $\tau'(0) = 1$, $\tau''(0) < 0$ then there is an open subset S of N with the following properties:

(i) S contains some interval $(0, \delta)$, $(\delta > 0)$ of the real axis, 0 is a boundary point of S and the boundary of S is tangent to the negative real axis at 0;

(ii) $\tau(S) \subseteq S$;

(iii) if, for $z \in S$, we define $z_0 = z$ and $z_n = \tau(z_{n-1})$ for $n \geq 1$, then $z_n \to 0$ as $n \to \infty$. Moreover there is an analytic function g on S for which the asymptotic relation

$$1/z_n = an + b\log n - ag(z) + o(1) \tag{2.1}$$

is valid as $n \to \infty$, where a and b are constants depending only on τ. In addition g is strictly increasing on $(0, \delta)$, $g(x) \to -\infty$ as $x \to 0_+$ and the functional equation $g(\tau(z)) = g(z) - 1$ holds for all $z \in S$.

In particular, if $\tau(z) = \log(1+z)$ we can take $S = \mathbf{C} \setminus (-\infty, 0]$, and this gives a function g on S for which

$$g(\log(1+z)) = g(z) - 1 \quad \text{for all } z \in S.$$

If in addition we assume $|\operatorname{Im}(z)| < \pi$, then also

$$g(e^z - 1) = g(z) + 1,$$

giving a solution of (A_1) in this region.

These results can be applied to the case in which an entire function $\varphi(z) = z + \sum_1^\infty c_n z^{n+1}$ is given with $c_1 > 0$ and $c_n \geq 0$ for $n > 1$. The above construction can be carried through with $\tau = \varphi^{-1}$ and $a = c_1$, $b = (c_2 - c_1^2)/c_1$. We invert the relation (2.1), omitting the $o(1)$ term, and put $w = g(z)$ to obtain the definition

$$\begin{aligned} f(w) &= \lim_{n\to\infty} f_n(w) \\ &= \lim_{n\to\infty} \varphi^{[n]}\{1/(a(n-w) + b\log n)\}, \end{aligned} \tag{2.2}$$

where $\varphi^{[n]}$ denotes the nth iterate of φ. Then we have

Theorem [7]. *If $c_1 > 0$, $c_n \geq 0$ for $n > 1$, and in addition either* (i) $c_2 \neq c_1^2$ *or* (ii) $c_3 < c_1^3$, *then the limit in* (2.2) *exists for all $w \in C$, and defines an entire nonconstant solution of* (A_2).

For convenience we write

$$\alpha_n(w) = (a(n-w) + b\log n)^{-1},$$

so that

$$f_n(w) = \varphi^{[n]}(\alpha_n(w)).$$

The following consequences of this construction are immediate.

Proposition 1.

(i) *$f(x) > 0$, $f'(x) \geq 0$ for all $x \in \mathbf{R}$, and $f^{(k)}(0) \geq 0$ for $k > 1$.*
(ii) *If $\varphi' \neq 0$ in $\mathbf{C}$, then $f' \neq 0$ in $\mathbf{C}$.*

Proof. (i) For each real x and sufficiently large n, we have $\alpha_n(x) > 0$ so that $f_n(x) > 0$ and $f(x) \geq 0$; $f'(x) \geq 0$ by a similar argument. Hence if $f(x_0) = 0$ then $f = 0$ on $(-\infty, x_0)$ which contradicts the fact that f is entire and nonconstant.

(ii) From (2.2) we deduce that

$$f_n'(w) = \left[\prod_{m=0}^{n-1} \varphi'\{\varphi^{[m]}(\alpha_n(w))\}\right] \alpha_n'(w),$$

which is never zero if $\varphi' \neq 0$, Hence, by Hurwitz' theorem,

$$f'(w) = \lim_{n\to\infty} f_n'(w)$$

must either vanish identically, which is excluded, or never be zero which is what we require.

3. Mapping properties of f and g

From now on φ and τ will denote the special functions $e^z - 1$ and $\log(1+z)$ (principal value) respectively. We begin by describing the action of the function g defined by (2.1), on the set $S = \mathbf{C} \setminus (-\infty, 0]$: the constants a, b are now equal $1/2$, $-1/6$.

Define the sequence $(\beta_n)_{n\geq 0}$ by $\beta_0 = -\infty$, $\beta_1 = -1$, and generally, $\beta_n = \exp(\beta_{n-1}) - 1$ for $n \geq 1$; evidently $\beta_{n-1} < \beta_n < 0$ and $\beta_n \to 0$ as $n \to \infty$.

For $n \geq 1$ let I_n denote the open interval (β_{n-1}, β_n) as in Figure 1 (p. 614).

For $n \geq 1$ we have $I_n = \tau(I_{n+1})$ and *define* $I_n = \tau(I_{n+1})$ for $n < 1$. Thus $I_o = \tau(I_1) = \{x + iy \colon y = \pi\}$ and for $n < 0$ all I_n are analytic arcs in the first quadrant which extend to infinity in the positive real direction, as indicated by the dotted arcs in Figure 1. All these arcs are disjoint and, by Fatou's result, the successive images of any individual point converge to zero.

The definition (2.1), which we write in the form

$$g(z) = \lim_{n\to\infty} [n - (\log n)/3 - 2/\tau^{[n]}(z)],$$

enables us to describe the action of g on the upper half-plane

$$H^* = \{z \colon \operatorname{Im}(z) \geq 0,\ z \neq \beta_n \text{ for } n \geq 1\}.$$

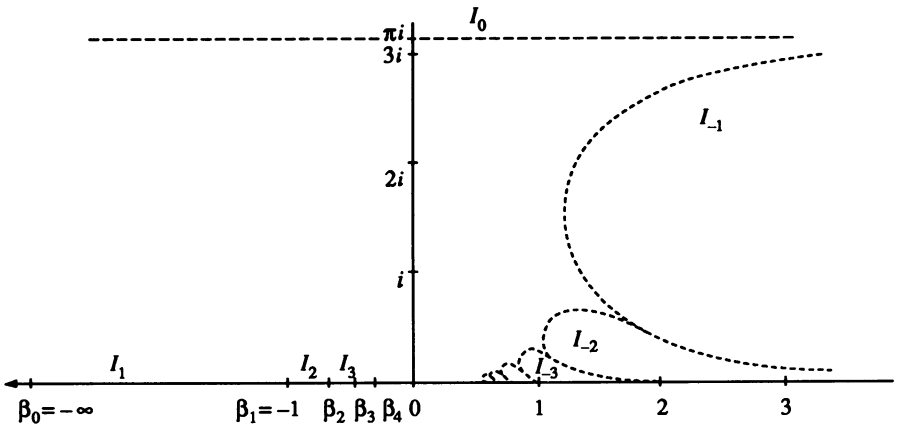

FIGURE 1

(We consider only the upper half plane: results for $\mathrm{Im}(z) \leq 0$ follow by symmetry.)

Consider first the effect of the mapping $z \to h(z) = -2/z$ on the loops I_n, $n \geq 0$. I_0 is mapped onto a circle, while, for $n \geq 1$, I_{-n} becomes a loop in the second quadrant, beginning and ending at 0, as indicated in Figure 2.

The translation $z \to z + n - (\log n)/3$ moves these loops to the right to begin and end at $n - (\log n)/3$; finally, taking the limit as $n \to \infty$ gives us a family of loops, $L_n = g(I_n)$, which begin and end at $+\infty$ and are invariant under

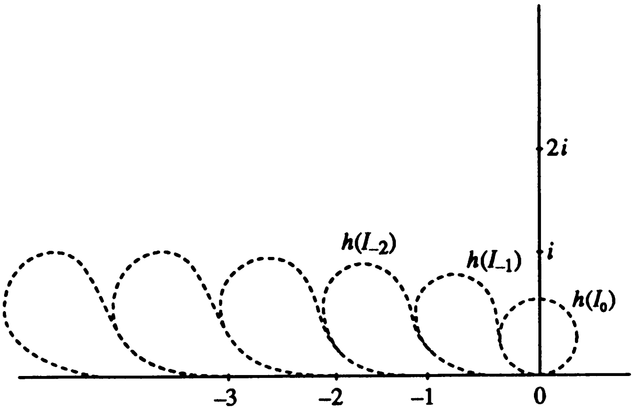

FIGURE 2

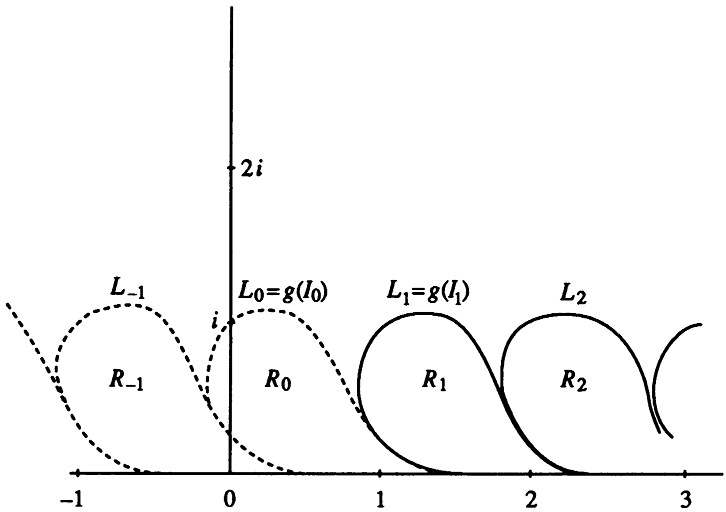

FIGURE 3

translation, i.e.,

$$\textit{for all } n, \quad L_{n+1} = L_n + 1 .$$

These loops are sketched in Figure 3.

We define R^n as the component of $\mathbf{C} \setminus L_n$ which is bounded in the direction of the imaginary axis and refer (imprecisely) to this set as the interior of L_n.

Proposition 2. (i) *There is a constant* C *such that if* $\operatorname{Re}(z) \geq 0$, $\operatorname{Im}(z) \geq 0$, *and* $|z| \geq e^{(e-1)} - 1$, *then*

$$|g'(z)| \leq C/\{|1+z||1+\tau(z)|\} .$$

(ii) *Let* $g(t+i\pi) = x(t) + iy(t)$, $x, y, t \in \mathbf{R}$, *be a parametrisation of the loop* L_0. *Then*

$$y(t) = O(1/\{f(x(t)-1)\}) \quad \textit{as } t \to \infty .$$

Note. It follows from part (ii) that each loop approaches the real axis faster than the reciprocal of any finite composition of exponential functions. In particular, each region $\mathbf{R}_n$, as well as the region between $\mathbf{R}_n$ and the real axis, has finite area.

Proof. (i) Recall that $z_n = \tau^{[n]}(z)$ for $n \geq 1$, so that $z_n \to 0$ as $n \to \infty$. Then let k be the least value of n with $|z_n| < e-1$; since $|z| > e^{(e-1)} - 1$, k is at least 2. Since $|z_{k-1}| \geq e-1$ we also have $|z_k| \geq 1$, and so z_k must lie in

$$D = \{z \colon \operatorname{Re}(z) \geq 0, \operatorname{Im}(z) \geq 0, \ 1 \leq |z| \leq e-1\} .$$

Let $C = \sup\{|g'(z)|: z \in D\}$. Then, from the relation $g(\tau(z)) = g(z) - 1$, we find that $g'(z) = g'(\tau(z))/(1+z)$, and hence

$$g'(z) = g'(z_n)\left\{\prod_{r=0}^{n-1}(1+z_r)\right\}^{-1}.$$

The result now follows by putting $n = k$, since each $|1 + z_r| \geq 1$.

(ii) Since

$$g(t + i\pi) = g(t) + i\int_0^\pi g'(t + iu)\,du,$$

we see that

$$x(t) = g(t) - \int_0^\pi \operatorname{Im}(g'(t + iu))\,du,$$

and

$$y(t) = \int_0^\pi \operatorname{Re}(g'(t + iu))\,du.$$

From this and part (i) it follows that both $|x(t) - g(t)|$ and $|y(t)|$ are bounded by $C\pi/(1+t)$. Now choose t_0 so that if $t \geq t_0$ then $C\pi/(1+t) \leq 1$, whence $g(t) \geq x(t) - 1$. So $t \geq f(x(t) - 1)$, and it follows that

$$|y(t)| \leq C\pi/\{1 + f(x(t) - 1)\},$$

as required. □

Now let H^0 be the open half plane $\{z: \operatorname{Im}(z) > 0\}$, which is mapped by g onto $H^0 \setminus \{\bigcup_{n=1}^\infty \overline{R}_n\} = K$ say. Points which lie above I_0 are mapped to R_0, and similarly for the regions enclosed by $(I_{-n})_{n\geq 1}$. Points in H^* near to β_n, $n \geq 1$, are mapped by g to points with large real part which lie between L_n and L_{n+1}. Points in H^* with large modulus are mapped to points with large real part which lie between L_1 and the real axis. The interval $(0, \infty)$ is mapped monotonically onto the whole real axis.

When we consider the entire function f which is inverse to g, we see that it maps each L_n to the corresponding I_n for all $n \in \mathbf{Z}$. So, for $n \geq 1$, L_n is a path on which $f(z) \to \beta_n$ as $|z| \to \infty$, and all β_n are asymptotic values. In addition, 0 is an asymptotic value, being $\lim_{x\to-\infty}(f(x))$. In the next section we shall show that there are no others.

4. Zeros and asymptotic values of f

We begin by determining the zeros of f. Since we know from Proposition 1 that f' is never zero these must all be simple.

Theorem 1. (i) *All zeros of f lie in the regions R_n, $n \geq 1$ (or their reflections in the real axis). In particular if*

$$c = \inf\{\operatorname{Re}(z): z \in L_1\}, \qquad d = \sup\{\operatorname{Im}(z): z \in L_1\},$$

then f has no zeros if $\operatorname{Re}(z) \leq c$, or if $|\operatorname{Im}(z)| \geq d$.

(ii) *The number u is a zero of f if and only if, for some $t \neq 0$, there is an integer $n \geq 1$ with $\varphi^{[n]}(t) = 0$ and $u = g(t) + n$.*

Note. From the functional equation it follows that $f(u) = 0$ implies $f(u+k) = 0$ for $k = 1, 2, \ldots$. (Evidently the same is true of any fixed point of φ.) Part (i) shows that these chains of zeros can go to infinity only in the positive direction, and part (ii) identifies their starting points.

Numerical estimates suggest the approximate values $c = 0.8$ and $d = 1.2$. The zeros of f with smallest real part are at $1.33 \pm i0.71 = g(\pm 2\pi i) + 1$.

Proof. Part (i) is immediate since f maps K to H^0.

To prove (ii) suppose that $u = g(t) + n$ and $\varphi^{[n]}(t) = 0$. Then $f(u) = f(g(t) + n) = \varphi^{[n]}(f(g(t)) = \varphi^{[n]}(t) = 0$.

Conversely suppose that $f(u) = 0$, and choose $n > 0$ so that $\operatorname{Re}(u - n) < c$. Then, for some $t \in H^*$, $g(t) = u - n$, $t = f(u - n)$, and so $0 = f(u) = f((u - n) + n) = \varphi^{[n]}(f(u - n)) = \varphi^{[n]}(t)$ as required. □

We next identify the asymptotic values of f.

Theorem 2. *The set of asymptotic values of f is $\{\beta_n\}_{n\geq 1} \cup \{0\}$.*

Proof. We have already shown that β_n and 0 are asymptotic values; it remains to show that there are no others.

Suppose that β is some complex number not equal to 0 or to any β_n, and that Γ is a continuous mapping of $[0, \infty)$ to C such that $|\Gamma(t)| \to \infty$ and $f(\Gamma(t)) \to \beta$ as $t \to \infty$.

Since $\beta \neq 0$, $\Gamma(t)$ must lie in the strip $|\operatorname{Im} z| \leq d$ found in Theorem 1(i), and so $\operatorname{Re}(\Gamma(t)) \to \infty$. Similarly, Γ cannot cross infinitely many of the loops L_n, $n \geq 1$ since this would also force $\beta = 0$. Hence we deduce that, for some $n = k$, say, $\Gamma(t)$ must got to infinity through the region between L_k and its reflection in the real axis. Since, in addition, β is not equal to any β_n, we can apply τ repeatedly without finding $\tau^{[n]}(\beta) = \infty$: doing this $k - 1$ times we find that the translation $\Gamma - (k - 1)$ is a path which tends to infinity through the region between L_1 and its reflection in the real axis, and on which f approaches $\tau^{[k-1]}(\beta)$. But the limit of f in this region is ∞, and this contradiction gives the required result. □

5. Property A and some open questions

The following property of entire functions is discussed by Edrei and Erdös [3].

Theorem 3A. *For any $A > 2$, the set $\{z: |f(z)| > A\}$ has finite area.*

We see that if $|f(z)| > A$, then

$$e^{\operatorname{Re} f(z-1)} = |f(z) + 1| > A - 1 > 1,$$

and so $\operatorname{Re} f(z - 1) > \log(A - 1) > 0$. Hence Theorem 3A is implied by

Theorem 3B. *If $B > 0$, the set $\{z: \operatorname{Re} f(z) > B\}$ has finite area.*

Proof. Recall the definition of $f(w)$ in §2 as $\lim_{n\to\infty} f_n(w)$, where $f_n(w) = \varphi^{[n]}(\alpha_n(w))$. We shall show that, for $n \geq 2$, the areas of the sets $W_n = \{w\colon \operatorname{Re} f_n(w) > B\}$ form a decreasing sequence, which is sufficient to establish the result.

We construct a sequence of sets $(A_n)_{n\geq 0}$ as follows.

Let $A_0 = \{z\colon \operatorname{Re} z > B\}$, and, for $n \geq 1$, let $A_{n,0} = \tau(A_{n-1})$. Then $A_{n,j} = A_{n,0} + 2j\pi i$ gives the image of A_{n-1} by the other branches of log, and we put $A_n = \bigcup_{j=-\infty}^{\infty} A_{n,j}$. (It is difficult to give an adequate idea of the sets A_n in a figure. For an indication of the intersection $\bigcap_{n=0}^{\infty} A_n$ where $B = 0$, see [2, Figure 2].) Note however that if $z \in A_n$ then $\operatorname{Re} z \geq b_n = \varphi^{[n]}(B) > 0$. From the construction we see that $\operatorname{Re} f_n(w) > B$ is equivalent to $f_n(w) \in A_0$ or $\alpha_n(w) \in A_n$, and so $W_n = \alpha_n^{-1}(A_n)$. Since, for $z \in A_n$, $\operatorname{Re} z \geq b_n$ and $\alpha_n^{-1}(t) = n - (\log n)/3 - 1/t$, it follows that W_n is a bounded set and has finite area.

Hence

$$\begin{aligned} m(W_n) &= \int_{W_n} dm(z\text{æ}\hat{p} \oint_{A_n} |\bar{z}|^2 \quad dm(z) \\ &= \int_{A_{n,o}} \sum_{-\infty}^{\infty} |z + 2j\pi i|^{-2}\, dm(z) \\ &= \int_{A_{n,0}} h(z)\, dm(z), \end{aligned}$$

say, where m denotes Lebesgue measure.

For $h(z) = \sum_{-\infty}^{\infty} |z + 2j\pi i|^{-2}$ we also have the expression

$$h(z) = (1 - e^{-2x})/\{2x(1 - e^{iy-x})(1 - e^{-iy-x})\},$$

which is obtained from the Poisson summation formula.

Since $A_{n,0} = \tau(A_{n-1})$, we deduce that

$$W_n = \int_{A_{n,0}} h(z)\, dm(z) = \int_{A_{n-1}} h(\tau(z))|\tau'(z)|^2\, dm(z).$$

If we compare this with

$$W_{n-1} = \int_{A_{n-1}} |z|^{-2}\, dm(z),$$

we see that, for our result, it is sufficient to show that, for $z \in A_{n-1}$,

$$h(\tau(z))|1 + z|^{-2} \leq |z|^{-2}.$$

By putting $w = \tau(z)$, $w \in A_n$, it follows that this is equivalent to

$$h(w) \leq |e^w/(e^w - 1)|^2 \quad \text{for } w \in A_n.$$

But this is valid for all w, $\operatorname{Re} w > 0$, since, on the left, we have

$$(1 - e^{-2x})/\{2x(1 - e^{iy-x})(1 - e^{-iy-x})\}$$

and, on the right,

$$|1-e^{-w}|^{-2} = 1/\{(1-e^{iy-x})(1-e^{-iy-x})\}.$$

Thus the desired inequality reduces to $1-e^{-2x} \le 2x$, which is immediate, and completes the proof of Theorems 3A and 3B. □

Finally we mention some open questions concerning the function which we have constructed.

(A) The convergence to the limit by which f is defined is slow: in fact it is shown in [7] that

$$f_n(w) = f(w) + O((\log n)/n).$$

However it appears (from our Figure 2 for instance) that convergence of the loops L_n to their limiting positions is very rapid. It would be interesting to find whether this is in fact the case.

(B) Is f admissible in the sense of Hayman [5]?

There are a number of properties which, in the case of a general mapping function φ, are inherited by f from φ: for instance

(i) if all $\varphi^{(n)}(0) \ge 0$ the same is true of f;

(ii) if φ is a "maximum modulus function," i.e., it satisfies, for all z, $|\varphi(z)| \le \varphi(|z|)$, then the same is true of f; and

(iii) if $\varphi' \ne 0$ in $\mathbf{C}$, then the same is true of f.

In this list, (i) is the result from [7] already quoted in §2, and (ii) can be proved in identical fashion, while (iii) is our Proposition 1(ii).

In view of this we make the stronger conjecture that f is admissible whenever φ is.

(C) There is some recent use in numerical analysis (see for instance [1], and other references there) of solutions of the Abel equation

$$F(x+1) = e^{F(x)}. \tag{5.1}$$

One represents a (large) real number x by $G(x) = n + t$, where n, t are defined by

$$\log^{[n]}(x) = t \in [0, 1);$$

here n is called the level and t the index of x.

The $F = G^{-1}$ is a solution of (5.1) which is once but not twice differentiable on $\mathbf{R}$.

However it is notoriously more difficult to construct real-analytic solutions of (5.1) than of our

$$f(x+1) = e^{f(x)} - 1, \tag{5.2}$$

due to the absence of a fixed point of the mapping $z \to e^z$ on the real axis.

It would be interesting to test the effect of using our solution of (5.2) in place of F: this would give a real-analytic representation of x by $g(x)$, with g

defined as in §2. The level of x would be the least k for which $f_k \le x < f_{k+1}$, where the sequence $(f_n)_{n\ge 0}$ is defined by

$$f_0 = 1, \qquad f_{n+1} = \exp(f_n) - 1, \quad \text{for } n \ge 0,$$

and the sequence (f_n) could be defined for negative values of n by the same relation if required.

(D) Referring to Theorem 3B, we could ask whether the set $\{z\colon \operatorname{Re} f(z) > 0\}$ has finite area. This seems unlikely since all the sets $\{z\colon \operatorname{Re} f_n(z) > 0\}$ have infinite area, though this of course is not conclusive.

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