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STATES ON W^*-ALGEBRAS AND ORTHOGONAL VECTOR MEASURES

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ABSTRACT. We show that every state on a W^*-algebra $\mathscr{A}$ without type I_2 direct summand is induced by an orthogonal vector measure on $\mathscr{A}$. This result may find an application in quantum stochastics [1, 7]. Particularly, it allows us to find a simple formula for the transition probability between two states on $\mathscr{A}$ [3, 8].

1. PRELIMINARIES

Here we fix some notation and recall basic facts about Hilbert–Schmidt operators (for details, see [9]). Let H be a complex Hilbert space and let $\mathscr{B}(H)$ denote the set of all linear bounded operators on H. An operator $A \in \mathscr{B}(H)$ is said to be a *trace class operator* if the series $\sum_{\alpha\in I}(Ae_\alpha, e_\alpha)$ is absolutely convergent for an orthonormal basis $(e_\alpha)_{\alpha\in I}$ of H. In this case the sum $\sum_{\alpha\in I}(Ae_\alpha, e_\alpha)$ does not depend on the choice of the basis $(e_\alpha)_{\alpha\in I}$ and is called a *trace* of A (abbreviated $\operatorname{Tr} A$). Let us denote by $C_1(H)$ the set of all trace class operators on H. Then $\operatorname{Tr} AB = \operatorname{Tr} BA$ whenever the product AB belongs to $C_1(H)$.

Suppose that $A \in \mathscr{B}(H)$. Then A is called a *Hilbert–Schmidt operator* if $\|A\|_2 = \{\sum_{\alpha\in I}\|Ae_\alpha\|^2\}^{1/2} < \infty$ for some (and therefore for any) orthonormal basis $(e_\alpha)_{\alpha\in I}$ of H. Let us denote by $C_2(H)$ the set of all Hilbert–Schmidt operators on H. Thus, $C_2(H)$ is a two-sided ideal in $\mathscr{B}(H)$, and $C_1(H) \subset C_2(H)$. Furthermore, the space $C_2(H)$ endowed with an inner product $\langle\cdot,\cdot\rangle$ given by the equality $\langle A, B\rangle = \operatorname{Tr} AB^*(A, B \in C_2(H))$ is a Hilbert space isomorphic to $l^2(E \times E)$, where E is an orthonormal basis of H.

2. ORTHOGONAL VECTOR MEASURES AND STATES ON A W^*-ALGEBRA

Let $\mathscr{A}$ be a W^*-algebra acting on a Hilbert space H, and let $\mathscr{P}(\mathscr{A})$ denote the lattice of all projections in $\mathscr{A}$. Our principal definition is as follows (for $\mathscr{A} = \mathscr{B}(H)$, see [7, 8]).

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Definition. Let $\mathscr{A}$ be a W^*-algebra acting on H. A mapping $m : \mathscr{P}(\mathscr{A}) \to H$ is called an orthogonal vector measure if for any set $(P_\alpha)_{\alpha\in I}$ of mutually orthogonal projections the following two conditions are satisfied:

(i) the set $(m(P_\alpha))_{\alpha\in I}$ is orthogonal in H, and
(ii) we have $m(\sum_{\alpha\in I} P_\alpha) = \sum_{\alpha\in I} m(P_\alpha)$, where the series on the right side converges in the norm topology on H.

Let $m : \mathscr{P}(\mathscr{A}) \to H$ be an orthogonal vector measure. Put, for any $P \in \mathscr{P}(\mathscr{A})$, $s_m(P) = \|m(P)\|^2$. Then $s_m : \mathscr{P}(\mathscr{A}) \to R$ is a state on $\mathscr{P}(\mathscr{A})$. Indeed, since m is bounded ($\|m(P)\|^2 \leq \|m(P)\|^2 + \|m(P^\perp)\|^2 = \|m(I)\|^2$), we see that s_m is bounded, too, and the equality $s_m(\sum_{\alpha\in I} P_\alpha) = \sum_{\alpha\in I} s_m(P_\alpha)$ follows immediately from the definition of m. The objective of this paper is to prove that every state may arise this way.

Theorem. *Let $s : \mathscr{P}(\mathscr{A}) \to R$ be a state on $\mathscr{P}(\mathscr{A})$, where $\mathscr{A}$ is a W^*-algebra without type I_2 direct summand such that $\mathscr{A}$ acts on a Hilbert space H with* $\dim H = \infty$. *Then there is an orthogonal vector measure $m : \mathscr{P}(\mathscr{A}) \to H$ such that $s(P) = \|m(P)\|^2$ for every $P \in \mathscr{P}(\mathscr{A})$.*

Proof. According to the Gleason–Christensen–Yeadon theorem [2, 6, 10], the state s on $\mathscr{P}(\mathscr{A})$ can be extended to a positive linear functional f on $\mathscr{A}$. Because f is completely additive, it has to be normal, and therefore we have the tracial formula

$$f(A) = \operatorname{Tr} BA \ (A \in \mathscr{A}),$$

where $B \in \mathscr{A}$ is a uniquely defined self-adjoint nonnegative trace class operator [10]. Then for any orthonormal basis $(e_\alpha)_{\alpha\in I}$ of H, we have

$$\sum_{\alpha\in I} \|B^{1/2} e_\alpha\|^2 = \sum_{\alpha\in I} (B^{1/2} e_\alpha, B^{1/2} e_\alpha) = \sum_{\alpha\in I} (B e_\alpha, e_\alpha) = \operatorname{Tr} B < \infty .$$

Thus, $B^{1/2} \in C_2(H)$. Making use of the fact that $C_2(H)$ is an ideal in $\mathscr{B}(H)$, define a mapping $\overline{m} : \mathscr{P}(\mathscr{A}) \to C_2(H)$ by the formula

$$\overline{m}(P) = B^{1/2} P \ (P \in \mathscr{P}(\mathscr{A})).$$

We shall prove that $\overline{m}$ is an orthogonal vector measure on $\mathscr{P}(\mathscr{A})$. Let $(P_\alpha)_{\alpha\in I}$ be a family of mutually orthogonal projections in $\mathscr{A}$. Put $P = \sum_{\alpha\in I} P_\alpha$. We have to show that $\overline{m}(P) = \sum_{\alpha\in I} \overline{m}(P_\alpha)$ (in $C_2(H)$). Let Y_α be an orthonormal basis of the space P_α. (By a harmless abuse of notation, let us identify the projection operators with the subspaces onto which they project.) Then $\bigcup_{\alpha\in I} Y_\alpha = \{v_\beta\}_{\beta\in J}$ is an orthonormal basis of P. Choose an arbitrary positive ε. Since we have $\|B^{1/2}P\|_2^2 = \sum_{\beta\in J} \|B^{1/2} P v_\beta\|^2 = \sum_{\beta\in J} \|B^{1/2} v_\beta\|^2 < \infty$, there is a finite subset J_0 of J such that $\sum_{\beta\in J\setminus J_0} \|B^{1/2} v_\beta\|^2 < \varepsilon$. Put $I_0 = \{\alpha \in I | Y_\alpha \cap \{v_\beta\}_{\beta\in J_0} \neq \emptyset\}$. Let $I_1 \supset I_0$ be a finite subset of I and let

$J_1 = \{\beta \in J | v_\beta \in \bigcup_{\alpha\in I_1} Y_\alpha\}$. If we put $Q = \sum_{\alpha\in I_1} P_\alpha$, we have

$$\begin{aligned}\left\|\overline{m}(P) - \sum_{\alpha\in I_1} \overline{m}(P_\alpha)\right\|_2^2 &= \|\overline{m}(P) - \overline{m}(Q)\|_2^2 \\ &= \sum_{\beta\in J} \|(B^{1/2}P - B^{1/2}Q)v_\beta\|^2 \\ &= \sum_{\beta\in J\setminus J_1} \|B^{1/2}v_\beta\|^2 \\ &\le \sum_{\beta\in J\setminus J_0} \|B^{1/2}v_\beta\|^2 < \varepsilon.\end{aligned}$$

Thus, $\overline{m}(\sum_{\alpha\in I} P_\alpha) = \sum_{\alpha\in I} \overline{m}(P_\alpha)$. Moreover, if $\alpha_1, \alpha_2 \in I$ and $\alpha_1 \ne \alpha_2$, then simple computations give

$$\langle \overline{m}(P_{\alpha_1}), \overline{m}(P_{\alpha_2})\rangle = \operatorname{Tr} B^{1/2} P_{\alpha_1} P_{\alpha_2} B^{1/2} = 0.$$

We therefore see that $\overline{m}$ is an orthogonal vector measure.

If $P \in \mathscr{P}(\mathscr{A})$, then $\|\overline{m}(P)\|^2 = \operatorname{Tr} B^{1/2}PB^{1/2} = \operatorname{Tr} BP = s(P)$. Thus, s is induced by $\overline{m}$. Finally, since there is a unitary mapping $u : C_2(H) \to H$, it remains only to set $m = u \circ \overline{m}$. The proof is complete.

Remark. Observe that our result is no longer valid if we admit $\dim H < \infty$. For instance, if $\mathscr{A} = \mathscr{B}(H)$ for finite-dimensional H, then the states on $\mathscr{P}(\mathscr{A})$ induced by orthogonal vector measures are exactly pure states [5]. Thus, if $\dim H < \infty$ and $\mathscr{A} = \mathscr{B}(H)$, then the theorem holds precisely for pure states on $\mathscr{P}(\mathscr{A})$.

Let us note in conclusion an explicit corollary of our kernel theorem. Let s, v be two states on the projection lattice $\mathscr{P}(\mathscr{A})$ of a W^*-algebra $\mathscr{A}$ without type I_2 direct summand. By our theorem, there are orthogonal vector measures m, t such that $\|m(P)\|^2 = s(P)$ and $\|t(P)\|^2 = v(P)$ for any $P \in \mathscr{P}(\mathscr{A})$. Following [8], we can put

$$P_{s,v} = \inf_{P\in\mathscr{P}(\mathscr{A})} [|\langle m(P), t(P)\rangle|^2 / \|m(P)\|^2 \cdot \|t(P)\|^2]$$

and interpret $P_{s,v}$ as a generalized transition probability between s and v given by the "vector states" m and t. We believe that this formula presents a considerable simplification of the previous ones (see [3, 4, 8] for a more detailed discussion).

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