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ON CONTRACTIONS WITHOUT DISJOINT INVARIANT SUBSPACES

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ABSTRACT. Assume T is a contraction with the following property: there exists an operator X with dense range such that $XT = WX$ where W is a bilateral shift. We give a necessary and sufficient condition that T has no disjoint invariant subspaces.

In [3] and [4], Olin and Thomson investigated subnormal operators without disjoint invariant subspaces. They called a (bounded linear) operator T on a Hilbert space *cellular-indecomposable* if the intersection of any two nonzero invariant subspaces of T is nonzero. In this note we extend the result for cellular-indecomposable subnormal contractions proved in [4].

Let T be a contraction on a separable Hilbert space $\mathscr{H}$, and suppose that T is not of class C_0; that is, $\lim_{n\to\infty} \|T^n x\| \neq 0$ for some $x \in \mathscr{H}$. It follows that there exist an isometry V and an operator X with dense range such that $XT = VX$ (see [5, Proposition II.3.5]). Kerchy [2] proved that if this V is a bilateral shift, then T has a nontrivial invariant subspace. (Note that if V is not unitary, then it can be replaced by a bilateral shift.)

Theorem. *Let T be a contraction on $\mathscr{H}$ and assume that there exists an operator X with dense range such that $XT = WX$ where W is the bilateral shift on L^2. Then T is cellular-indecomposable if and only if there exists a quasiaffinity (i.e. an injection with dense range) Y such that $YT = SY$ where S is the unilateral shift on H^2.*

The existence of the operator X in the theorem is equivalent to the condition that $\Theta_T(\zeta)^*$ is not isometric a.e. on the unit circle $\mathbb{T}$ where Θ_T is the characteristic function of T (see [7]). If T is a contraction such that $YT = SY$ where Y is a quasiaffinity and S is the unilateral shift, then the restriction of T to any of its cyclic invariant subspaces is quasisimilar to S (see [1, Corollary 15]). Therefore our theorem extends the result of [4]. Such contractions were also considered in [8].

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Proof of theorem. Assume that there exists a quasiaffinity Y such that $YT = SY$. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two nonzero invariant subspaces of T. For $i = 1, 2$, there exists an injection K_i such that $K_iS = TK_i$ and $\operatorname{ran} K_i \subseteq \mathscr{M}_i$ (see [1, Lemma 3]). Since YK_i commutes with S, we have $YK_i = f_i(S)$ for a nonzero function $f_i \in H^\infty$. (For a contraction A whose unitary part is absolutely continuous or acts on the space $\{0\}$ and $f \in H^\infty$, $f(A)$ is an operator obtained by the Sz.-Nagy and Foias functional calculus [5].) It follows from the injectivity of Y that $Y(\mathscr{M}_1 \cap \mathscr{M}_2) = Y\mathscr{M}_1 \cap Y\mathscr{M}_2$. Therefore we have

$$\begin{aligned} Y(\mathscr{M}_1 \cap \mathscr{M}_2) &\supseteq YK_1H^2 \cap YK_2H^2 = f_1H^2 \cap f_2H^2 \\ &\supseteq f_1f_2H^2 \neq \{0\}, \end{aligned}$$

and $\mathscr{M}_1 \cap \mathscr{M}_2 \neq \{0\}$.

Conversely, we assume that T is cellular-indecomposable. Obviously T is completely nonunitary. By the proof of the theorem in [7] (see [2] for a special case), it follows from the existence of the operator X that there are an operator Z and a vector x_0 satisfying the following conditions:

(1) $ZT = WZ$ and

(2) the function $\log|Zx_0|$ is integrable.

The condition (2) implies $Zx_0 = ug$ where $|u| = 1$ a.e. and g is an outer function in H^2, and so $W|(Z\mathscr{M}_0)^-$ is a unilateral shift of multiplicity one where $\mathscr{M}_0 = \bigvee_{n\geq 0} T^n x_0$. Since $Z|\mathscr{M}_0$ is injective (see the proof of [1, Corollary 15]), $\ker Z \cap \mathscr{M}_0 = \{0\}$. Since the subspaces $\ker Z$ and $\mathscr{M}_0$ are invariant under T and T is cellular-indecomposable, it follows that $\ker Z = \{0\}$.

We claim that for every $x \in \mathscr{H}$, there exists a nonzero $f \in H^\infty$ such that $Zf(T)x \in (Z\mathscr{M}_0)^-$. For this purpose, take $x \in \mathscr{H}$ and let $\alpha = \{\zeta \in \mathbb{T} : |(Zx)(\zeta)| \geq |(Zx_0)(\zeta)|\}$. Let us choose a function $h \in H^\infty$ such that $|h(\zeta)| = 1/2$ for $\zeta \in \alpha$ and $|h(\zeta)| = 2$ for $\zeta \in \mathbb{T}\backslash\alpha$. Then on α we have

$$\begin{aligned} |Zx + hZx_0| &\geq |Zx| - |h|\,|Zx_0| \\ &\geq (1 - |h|)|Zx_0| = (1/2)|Zx_0|, \end{aligned}$$

and on $\mathbb{T}\backslash\alpha$ we have

$$\begin{aligned} |Zx + hZx_0| &\geq |h|\,|Zx_0| - |Zx| \\ &\geq (|h| - 1)|Zx_0| = |Zx_0|. \end{aligned}$$

Thus $\log|Zx + hZx_0|$ is integrable, and so $Zx + hZx_0 = vk$ where $|v| = 1$ a.e. and k is outer. Then, setting $\mathscr{M} = \bigvee_{n\geq 0} T^n(x + h(T)x_0)$, we have

$$uH^2 \cap vH^2 \supseteq Z\mathscr{M}_0 \cap Z\mathscr{M} \supseteq Z(\mathscr{M}_0 \cap \mathscr{M}) \neq \{0\}$$

because T is cellular-indecomposable. Thus there exists a nonzero function $f \in H^\infty$ such that $\overline{u}vf \in H^\infty$. Then we have

$$\begin{aligned} Zf(T)x &= fZx = f(vk - hZx_0) \\ &= u(\overline{u}vfk - hfg) \in uH^2 = (Z\mathscr{M}_0)^-. \end{aligned}$$

This establishes the claim.

Let $\mathscr{N} = \{x \in \mathscr{H} : Zx \in (Z\mathscr{M}_0)^-\}$, which is an invariant subspace of T. Let $T_1 = PT|\mathscr{H} \ominus \mathscr{N}$ where P is the orthogonal projection onto $\mathscr{H} \ominus \mathscr{N}$. The claim given above implies that for every $x \in \mathscr{H} \ominus \mathscr{N}$, there exists a nonzero function $f \in H^\infty$ such that $f(T_1)x = 0$, and so T_1 is of class C_0 by [6]. That is, there exists an inner function q such that $q(T_1) = 0$. Then $q(W)Z\mathscr{H} \subseteq (Z\mathscr{M}_0)^-$. Therefore $W|(q(W)Z\mathscr{H})^-$ is unitarily equivalent to the unilateral shift S. Since $q(W)Z$ is injective, it follows that there is a quasiaffinity Y such that $YT = SY$.

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