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RECURRENT HOMEOMORPHISMS ON $\mathbb{R}^2$ ARE PERIODIC

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ABSTRACT. A homeomorphism $f: (X, d) \to (X, d)$ of a metric space (X, d) onto X is *recurrent* provided that for each $\varepsilon > 0$ there exists a positive integer n such that f^n is ε-close to the identity map on X. The notion of a recurrent homeomorphism is weaker than that of an almost periodic homeomorphism. The result announced in the title generalizes the theorem of Brechner for almost periodic homeomorphisms and answers a question of R. D. Edwards.

1. INTRODUCTION

Let (X, d) be a locally compact metric space. Let id_X denote the identity function on X, and let Z (resp. Z^+) denote the set of integers (resp. non-negative integers). A homeomorphism $g: X \to X$ of X onto X is *almost periodic* if for each $\varepsilon > 0$ there exists a relatively dense set A in Z (i.e., there exists $N \in Z^+$ such that $[n, n+N] \cap A \neq \varnothing$ for each $n \in Z$) such that $d(g^m, \mathrm{id}_X) < \varepsilon$ for each $m \in A$.

A homeomorphism $g: X \to X$ of X onto X is *recurrent* if for each $\varepsilon > 0$ there exists $n > 0$ such that $d(g^n, \mathrm{id}_X) < \varepsilon$.

For X compact the following are equivalent [Got]:

(1) g is almost periodic.
(2) $\{g^n | n \in Z\}$ is equicontinuous.
(3) $\{g^n | n \in Z\}$ has compact closure in the space of all homeomorphisms of X onto X with compact open topology.

Clearly, periodic homeomorphisms are almost periodic and almost periodic homeomorphisms are recurrent. None of these implications can be reversed. By [Bre], almost periodic homeomorphisms of the plane $\mathbb{R}^2$ with the usual metric d are periodic. Hence, each almost periodic homeomorphism of the

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plane is conjugate either to a rational rotation or to a reflection about a line [Got], [Eil].

The main purpose of this paper is to prove that for the plane $\mathbb{R}^2$ with its usual metric d recurrent homeomorphisms are periodic. This answers a recent question of R. D. Edwards. This result was claimed in [Hac-1], but the proof given there appears to be deficient [Hac-2]. Theorem 1 gives a positive solution to a problem raised by J. Hachigian for $n = 2$. The case $n = 1$ was done in [Coh-Hac]. We are indebted to Professor Morton Brown for references [Hac-1], [Hac-2], and [Coh-Hac] and for suggesting to us the term "recurrent homeomorphism."

2. The main result

In this section we will prove the main result of the paper, but first we prove a special case of the main result under somewhat weaker hypotheses.

By a *domain* we will mean a nonempty, bounded, connected, simply connected, open subset of $\mathbb{R}^2$. We denote the closure (resp. boundary) of a set A by $\mathrm{Cl}(A)$ (resp. $\mathrm{Bd}(A)$). We let $B(A, \varepsilon) = \{x \in X | d(x, A) < \varepsilon\}$.

Let $h: X \to X$ be a homeomorphism. A set A in X is *h-invariant* if $h(A) \subset A$, and A is *completely h-invariant* if $h(A) = A$. A homeomorphism $h: X \to X$ of X onto X is *arc-recurrent* (resp. *point-recurrent*) provided that for each arc A (resp. for each point p) in X and for each $\varepsilon > 0$ there exists a positive integer n such that $h^n(A) \subset B(A, \varepsilon)$ (resp. $d(h^n(p), p) < \varepsilon$). Clearly, each recurrent homeomorphism is arc-recurrent and each arc-recurrent homeomorphism is point-recurrent. Notice that each irrational rotation of $\mathbb{R}^2$ is arc-recurrent but not recurrent. For domains we can prove the following:

Theorem 1. *Let U be a domain and let h: $\mathrm{Cl}(U) \to \mathrm{Cl}(U)$ be an arc-recurrent homeomorphism such that $h|\mathrm{Bd}(U) = \mathrm{id}_{\mathrm{Bd}(U)}$. Then $h = \mathrm{id}_{\mathrm{Cl}(U)}$.*

Proof. Note first that h is orientation preserving. Let $\mathrm{Fix}(h) = \{x \in \mathrm{Cl}(U) | h(x) = x\}$. Suppose $\mathrm{Fix}(h) \neq \mathrm{Cl}(U)$, or there is nothing to prove. Then $\mathrm{Bd}(U) \subset \mathrm{Fix}(h)$ and $\mathrm{Fix}(h)$ is closed. We prove first that $\mathrm{Fix}(h)$ is not connected. If $\mathrm{Fix}(h)$ were connected, let W be a component of $U \setminus \mathrm{Fix}(h)$. Then W would be homeomorphic to $\mathbb{R}^2$ and $h|W$ would be an orientation preserving fixed-point free homeomorphism onto W [Bro-Kis]. By a theorem of Brouwer [And], $\mathrm{Lim}_{n\to\infty} \sup h^n(x) \in \mathrm{Bd}(W)$ for $x \in W$. This would contradict point-recurrence. Thus, there exists a component F' of $\mathrm{Fix}(h)$ such that $U \setminus \mathrm{Fix}(h)$ separates $\mathrm{Bd}(U)$ from F'. Let F be the topological hull of F' (i.e., F is the union of F' together with all of the bounded components of $\mathbb{R}^2 \setminus F'$).

By [Bro-Kis], $h(F) = F$. Since h is one to one, $h(U \setminus F) = U \setminus F$.

Let T_1 be a simple closed curve in $U \setminus \mathrm{Fix}(h)$ which separates F from $\mathrm{Bd}(U)$. Let V_1 be the component of $\mathrm{Cl}(U) \setminus T_1$ which meets $\mathrm{Bd}(U)$. Let

$$A = \{x \in \mathrm{Cl}(U) | h^n(x) \notin V_1 \text{ for } n \in Z^+\}.$$

Then A is a closed, h-invariant set. Since h is point-recurrent, A is completely

h-invariant. So $A \subset \mathrm{Cl}(U)\backslash V_1$ is a compact set and $F \subset A$. Let C be the component of F in A. Since $C \cap \mathrm{Fix}(h) \neq \varnothing$, C is invariant. Also, by the same argument as was used in the construction of F, C does not separate the plane.

Claim 1. $F \neq C$.

Proof of Claim 1. Suppose $C = F$. Since the components of a compact Hausdorff space are quasi-components [Eng, p. 438], there exists a simple closed curve T_2 in $U\backslash A$ which separates T_1 and F. Let V_2 be the component of $\mathrm{Cl}(U)\backslash T_2$ which contains V_1. Let

$$H = \bigcup_{n=0}^{\infty} h^n(\mathrm{Cl}(V_2)) \subset \mathrm{Cl}(U)\backslash F.$$

Now, H is h-invariant and $H \cap C = 0$. If H is compact, then H is closed, and hence H is completely h-invariant, since h is point-recurrent. Hence, $\mathrm{Cl}(U)\backslash H \subset \mathrm{Cl}(U)\backslash V_2$ is h-invariant. So $\mathrm{Cl}(U)\backslash H \subset A$. The Boundary Bumping Theorem [Eng, p. 439] states that if G is a proper open subset of a continuum M, and N is a component of G, then $\mathrm{Cl}(G)$ meets $\mathrm{Bd}(G)$. Since C is a component of $\mathrm{Cl}(U)\backslash H$, C meets $\mathrm{Bd}(\mathrm{Cl}(U)\backslash H)$, which is a contradiction. Hence, H is not compact.

By [Hom-Kin], $\{h^n(\mathrm{Cl}(V_2))\}_{n=0}^{\infty}$ is a bulging sequence (i.e., $h^n(\mathrm{Cl}(V_2))\backslash \bigcup_{i=0}^{n-1}(\mathrm{Cl}(V_2)) \neq \varnothing$ for $n \in Z^+$), and there exists a point $x \in \mathrm{Cl}(V_2)$ such that $h^n(x) \notin V_2$ for each $n \in Z^+$. Since h is point-recurrent, $x \in \mathrm{Cl}(V_2) \cap (\mathrm{Cl}(U)\backslash V_2) = T_2$. Clearly, $x \in A$. This contradicts the fact that $T_2 \cap A = \varnothing$. The claim is proved.

Now, $U\backslash C$ is homeomorphic to the open annulus

$$Y = \{x \in \mathbb{R}^2 | 1 < |x| < 2\},$$

since C is a continuum in the domain U which does not separate the plane. Hence [Eps] there is a uniformization $\varphi\colon U\backslash C \to Y$, i.e., φ is a homeomorphism of $U\backslash C$ onto Y which maps crosscuts onto crosscuts. (A *crosscut* K of $U\backslash C$ is an arc in $\mathrm{Cl}(U\backslash C)$ such that $K \cap \mathrm{Bd}(U\backslash C)$ is the set of endpoints of K and these endpoints lie in one component of $\mathrm{Bd}(U\backslash C)$.) We may suppose φ maps the points of $U\backslash C$ near C to points of Y near $S_1 = \{x \in \mathbb{R}^2 | \, |x| = 1\}$ and φ maps the points of $U\backslash C$ near $\mathrm{Bd}(U)$ to points of Y near $S_2 = \{x \in \mathbb{R}^2 | \, |x| = 2\}$.

Let $g = \varphi \circ h \circ \varphi^{-1}\colon Y \to Y$. Then g is a homeomorphism of Y onto Y, and since h is orientation preserving, g is also orientation preserving. By [Eps], g extends to an orientation preserving homeomorphism $G\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(Y)$. Since $h|\,\mathrm{Bd}(U) = \mathrm{id}_{\mathrm{Bd}(U)}$ and $\mathrm{Bd}(C)\backslash \mathrm{Fix}(h) \neq \varnothing$, we have $G|S_2 = \mathrm{id}_{S_2}$ and $G|S_1 \neq \mathrm{id}_{S_1}$.

Choose an arc I in $\mathrm{Cl}(U)$ such that I irreducibly joins $\mathrm{Bd}(U)$ to C, $I \cap \mathrm{Bd}(U) = \{a\}$, $I \cap C = \{b\}$, and $h(b) \neq b$. Then there exist points $\alpha \in S_2$

and $\beta \in S_1$ such that $J = \{\alpha, \beta\} \cup \varphi(I\backslash\{a, b\})$ is an arc in $\mathrm{Cl}(Y)$ which is irreducible from S_1 to S_2. The points α and β are the endpoints of J, $G(\alpha) = \alpha$ and $G(\beta) \neq \beta$. Let $\varepsilon > 0$.

Claim 2. There is a positive integer n such that $G^n(J) \subset B(J, \varepsilon)$.

Proof of Claim 2. Choose sequences $\{L_i\}_i$ and $\{R_i\}_i$ of arcs in $\mathrm{Cl}(U)\backslash I$ converging to I such that

(1) for each i and j, I separates L_i from R_j in a connected neighborhood W of I in $(U\backslash C) \cup I$,
(2) each L_i and each R_i meets each of C and $\mathrm{Bd}(U)$ in exactly one point, and
(3) in W, L_{i+1} (resp. R_{i+1}) separates L_i (resp. R_i) from I for each $i \in Z^+$.

For each $i \in Z^+$, let L_i^0 (resp. R_i^0) be the arc L_i (resp. R_i) minus its endpoints. Then $\widehat{L}_i = \mathrm{Cl}(\varphi(L_i^0))$ (resp. $\widehat{R}_i = \mathrm{Cl}(\varphi(R_i^0))$) are arcs in $\mathrm{Cl}(Y)$ converging to J such that each $\widehat{L}_i$ and each $\widehat{R}_i$ intersects each of S_1 and S_2 in exactly one point.

Suppose that, for each positive integer n, $G^n(J)\backslash B(J, \varepsilon) \neq \varnothing$. Choose $i \in Z^+$ such that the component O of $\mathrm{Cl}(Y)\backslash(\widehat{L}_i \cup \widehat{R}_i)$ which contains J is in $B(J, \varepsilon)$, and $G(\beta) \notin O$. Then for each $n > 0$, $G^n(J) \cap \varphi(L_i^0 \cup R_i^0) \neq \varnothing$. Hence, $h^n(I) \cap (L_i \cup R_i) \neq \varnothing$ for each n. This contradicts the fact that there exists a positive integer n such that $h^n(I) \subset \mathrm{Cl}(U)\backslash(R_i \cup L_i)$. The claim is proved.

Let Z be the universal covering space of the closed annulus $\mathrm{Cl}(Y)$, let $\rho: Z \to \mathrm{Cl}(Y)$ be the covering projection, and let $\widetilde{G}: Z \to Z$ be a lifting of G. Then Z is the product of the line with an arc and $\mathrm{Bd}(Z) = Z_1 \cup Z_2$, where Z_1 and Z_2 are lines such that $\rho(Z_i) = S_i$ for $i = 1, 2$, $\widetilde{G}|Z_2 = \mathrm{id}_{Z_2}$ and $\widetilde{G}|Z_1 \neq \mathrm{id}_{Z_1}$. Assign a natural linear order to the line Z_1. Let $\widetilde{L}_i$, $\widetilde{J}$, and $\widetilde{R}_i$ be lifts of $\widehat{L}_i$, J an $\widehat{R}_i$, respectively, such that $\widetilde{O}$, the component of $\widetilde{J}$ in $Z\backslash(\widetilde{L}_i \cup \widetilde{R}_i)$, maps homeomorphically onto the set O defined in the proof of Claim 2. Let $\widetilde{\beta} \in \widetilde{J} \cap \rho^{-1}(\beta)$. Since $G(\beta) \notin O$, $\widetilde{G}(\widetilde{\beta}) \notin \widetilde{O}$. Without loss of generality, $\widetilde{\beta} < \widetilde{G}(\widetilde{\beta})$ in Z_1. Since G, hence $\widetilde{G}$, are orientation preserving homeomorphisms,

$$\widetilde{\beta} < \widetilde{G}(\widetilde{\beta}) < \cdots < \widetilde{G}^n(\widetilde{\beta}) \quad \text{for each } n \in Z^+.$$

Hence, for each $n \in Z^+\backslash\{0\}$, $\widetilde{G}^n(\widetilde{J}) \cap (\widetilde{L}_i \cup \widetilde{R}_i) \neq \varnothing$. This implies that $G^n(J) \cap (\widehat{L}_i \cap \widehat{R}_i) \neq \varnothing$ for each $n > 0$, which contradicts Claim 2. This completes the proof of the theorem.

Remark. The hypothesis that $h|\,\mathrm{Bd}(U) = \mathrm{id}_{\mathrm{Bd}(U)}$ in Theorem 1 can be replaced by the assumption that h is orientation preserving and h has at least one accessible fixed point on $\mathrm{Bd}(U)$.

Lemma 2. *Let* $h: X \to X$ *be a recurrent homeomorphism of the metric space* (X, d) *onto* X*, and let* n *be a positive integer. Then* h^n *is recurrent.*

Proof. Let $\varepsilon > 0$ be given. There is a positive integer k such that $d(h^k, \mathrm{id}_X) < \varepsilon/n$. Then, $d(h^{n\cdot k}, \mathrm{id}_X) < n \cdot \frac{\varepsilon}{n} = \varepsilon$.

Theorem 3. *If* $h: \mathbb{R}^2 \to \mathbb{R}^2$ *is a recurrent homeomorphism of the plane with its usual metric* d *onto* $\mathbb{R}^2$*, then* h *is periodic.*

Proof. Let n be a positive integer such that $d(h^n, \mathrm{id}_{\mathbb{R}^2}) < 1$. By [Bro-1], h^n is orientation preserving, and by Lemma 2, h^n is recurrent. Let D^0 be the open and D the closed unit ball centered at the origin. Define $\psi: \mathbb{R}^2 \to D^0$ by $\psi(re^{i\theta}) = \frac{r}{1+r}e^{i\theta}$ and $\varphi' = \psi \circ h^n \circ \psi^{-1}: D^0 \to D^0$. Since $d(h^n, \mathrm{id}_{\mathbb{R}^2}) < 1$, φ' extends to a homeomorphism $\varphi: D \to D$ such that $\varphi | \mathrm{Bd}(D) = \mathrm{id}_{\mathrm{Bd}(D)}$. Since $d(\varphi^m, \mathrm{id}_D) \leq d(h^{n\cdot m}, \mathrm{id}_{\mathbb{R}^2})$ and h^n is recurrent, φ is also recurrent. By Theorem 1, $\varphi = \mathrm{id}_D$. Hence, $h^n = \mathrm{id}_{\mathbb{R}^2}$.

Remark. Note that the hypothesis of Theorem 3 (i.e., h is recurrent) is used only to ensure that the induced map φ on the closed unit ball is arc-recurrent and the identity on its boundary. Hence the hypothesis of Theorem 3 can be weakened accordingly.

Added in proof. R. D. Edwards communicated to us that he proved Theorem 3 independently.

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