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TOTALLY REAL SETS IN C^2

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ABSTRACT. We establish the polynomial convexity of certain totally real disks and of annuli in the unit torus satisfying a topological condition.

1

Jöricke [1] recently proved that a totally real disk contained in the unit sphere in C^2 is polynomially convex. More precisely, the result of [1] involves analytic extension but, by work of Stout [7] and Lupacciolu [2], the polynomial convexity follows; see also Rosay and Stout [4]. In this note we shall prove the analogous and easier result when the sphere is replaced by the set $M = \{(z, w): |z| = 1\}$.

Theorem 1. *Let K be a smooth totally real compact disk contained in the real hypersurface M. Then K is polynomially convex.*

One could possibly prove this by closely imitating the argument of Jöricke; however, the approach we follow, although it has some of the elements of the proof of [1], is probably shorter. Just as in [1] this is not a local result as the unit torus sits in M as a totally real 2-manifold which is not polynomially convex.

Example. If we allow K to fail to be totally real at a single point then it may not be polynomially convex. A simple example for such K is the image of the unit disk by the map $z \to (\exp(i \cdot |z|^2), z)$. Then K, which is essentially the graph of the exponential, has a complex tangent only at the point $(1, 0)$ and is clearly not polynomially convex since it has circles as fibers over a subarc of the unit circle of the z-plane. An analogous example in the context of [1] is obtained from the map $z \to (z, \sqrt{1-|z|^2})$. It should be noted that Wermer (see [3, p. 34]) has given an example of a totally real disk in C^2 which is not polynomially convex.

For a set S in C^2 and $z \in C$ we denote the fiber $\{w \in C: (z, w) \in S\}$ by S_z. To prove the theorem we first claim that K_z is polynomially convex

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in C for all z. Suppose not. Then there is an a in the unit circle such that K_a is not polynomially convex and hence, by Runge's theorem, $C\backslash K_a$ is not connected. By Alexander duality [5, pp. 296, 334] $\check{H}^1(K_a, Z)$ is nontrivial. Now K is topologically a disk which we can assume sits in a copy E of R^2. Identifying K_a with $\{a\} \otimes K_a \subseteq K$ and applying Alexander duality again for $K_a \subseteq E$, we conclude that $E\backslash K_a$ is not connected.

Since K is totally real and M has real dimension 3 there is a well-defined real tangent line bundle on K given by the intersection of the complex tangent space of M with the tangent space of K. Since K is contractible, there is a unit tangent vector field v which is a section of this bundle. That is, v is a unit vector field on K which lies pointwise in the complex tangent space to M at each point of K; cf. [1]. Consider the integral curves of v in K. Since v is at each point a derivative in a w direction, the vector field v applied to the function z vanishes identically. Therefore, z is constant on each integral curve. If p is a point of K not in K_a then, by the Poincaré-Bendixson theory, the integral curve through p joins p to the boundary of K. Since z at p is different from a, this integral curve is disjoint from K_a. This implies that $E\backslash K_a$ is connected. This is a contradiction.

Thus each K_z is polynomially convex. Since $\check{H}^1(K, C) = 0$, it follows directly from a result of Stolzenberg [6, Corollary 2.20] that K is polynomially convex.

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The aforementioned result of Stolzenberg requires that the set K satisfy $\check{H}^1(K, C) = 0$. However the idea of his proof holds in more general cases, for example, in the following setting. Let T^2 be the unit torus in C^2. We identify the fundamental group of T^2 with Z^2 as follows: $[r, s] \in Z^2$ is identified with (the homotopy class of) the curve $\{(\exp(irt), \exp(ist)): 0 \leq t \leq 2\pi\}$. Let A be a compact annulus contained in T^2 and g a simple closed curve contained in A that generates the fundamental group of A. Let $[p, q]$ be homotopic in T^2 to g; $\pm[p, q]$ is independent of the choice of g.

Theorem 2. *If $pq < 0$ or $p = 0 = q$, then A is polynomially convex.*

Remark. If $pq > 0$ or exactly one of p and q is zero, then A need not be polynomially convex. Indeed the following is easily verified.

Lemma. *If g is a simple closed curve in T^2 which is not null-homotopic, then g is homotopic in T^2 to a curve $[p, q]$ with p and q relatively prime. In particular, if $q = 0$ then $p = 1$ or $p = -1$.*

By the lemma, we can assume that p and q are relatively prime. This implies that $\{(z, w) \in T^2: z^q = w^p\}$ is a (connected!) simple closed curve in T^2 which is not polynomially convex. Then a tubular neighborhood of this curve in T^2 provides an example of a nonpolynomially convex annulus A with $g = [p, q]$.

Proof of Theorem 2. If $pq < 0$, then by symmetry we can assume that $p < 0$ and $q > 0$. Let $a = -p$ and set $f(z, w) = z^q w^a$. Then f is identically 1 on the curve $[p, q]$. Since f has modulus 1 on T^2 and since g is homotopic to the curve $[p, q]$ in T^2, it follows that f restricted to A lifts to a map F of A into C such that $\exp \circ F = f$; i.e. F is a logarithm of f on A. Then F extends to be a logarithm of f on a neighborhood N of A in C^2.

We claim that $(\hat{A})_z = A_z$ for all z in the unit circle. Since $(T^2)_z$ is a peak set of T^2, it is enough to show that A_z is a proper subset of the unit circle. Suppose not. Then A contains a circle k of the form $(T^2)_z$ for some z. But then, as the fundamental group of A is singlely generated, it follows that k is homotopic in A to some multiple of g. Hence k is also homotopic in T^2 to a multiple of $[p, q]$; this is clearly false—a contradiction.

To prove that A is polynomially convex we argue by contradiction and suppose otherwise. For $r < 1$ we set $Q = \hat{A} \cap \{(z, w): r \leq |z| \leq 1\}$. If r is sufficiently close to 1, by the last paragraph, Q is contained in N. By the local maximum modulus principle (cf. [6]) the Shilov boundary of the algebra of functions on Q which are locally in $P(Q)$ is contained in the union of $Q \cap T^2 = A$ and $Q \cap \{(z, w): |z| = r\}$. On the first set f has modulus 1 and on the second set f has modulus $\leq r^q$. Since $F = \log(f)$ is locally in $P(Q)$, the boundary of $F(Q)$ is contained in the union of two sets: the vertical line $\{\mathrm{Re}(z) = 0\}$ and the set $\{\mathrm{Re}(z) \leq q \cdot \log(r)\}$. This implies that $F(Q)$ does not meet $\{z: q \cdot \log r < \mathrm{Re}(z) < 0\}$. Hence A is relatively open in $\hat{A}$. This implies that A is polynomially convex.

If $p = 0 = q$ then g is null-homotopic in T^2; hence g bounds a disk in T^2. Thus A is contained in a compact disk K in T^2 and the polynomial convexity of A follows, say by Theorem 1.

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