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INDEX OF FAITHFUL NORMAL CONDITIONAL EXPECTATIONS

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ABSTRACT. Let E be a faithful normal conditional expectation of a factor M onto its subfactor N, and the index of E be denoted by IND_E. We investigate the question: For two such faithful normal conditional expectations E_1, E_2 of M onto N, when does $\mathrm{IND}_{E_1} = \mathrm{IND}_{E_2}$ hold? In this paper we answer this question completely for type I factor M. We also derive a tensor product formula for index, i.e., $\mathrm{IND}_{E_1 \otimes E_2} = (\mathrm{IND}_{E_1})(\mathrm{IND}_{E_2})$. For any $\alpha > 9$ we construct uncountable nonisomorphic faithful normal conditional expectations E of a factor M onto its subfactor N with $\mathrm{IND}_E = \alpha$ such that both of M and N, are of type I or II or III_λ, $0 \le \lambda \le 1$. For each $\beta \in \{4\cos^2 \pi/n, |n \ge 3\} \cup [4, \infty)$ we exhibit a type III_λ factor M and its subfactor N and a faithful normal conditional expectation E such that $\mathrm{IND}_E = \beta$.

1. INTRODUCTION

In 1984 Kosaki generalized the definition of index to arbitrary factors M and their subfactors N in case of the existence of a faithful normal conditional expectation E of M onto N [6]. The index of E, denoted by IND_E, is defined as $E^{-1}(I)$. Then the natural question to ask is "for what pairs of E_1, E_2 do we have $\mathrm{IND}_{E_1} = \mathrm{IND}_{E_2}$?" In §2 of this article we answer this question completely for type I factors M (see 2.8). In general, we are able to exhibit uncountable nonouterconjugate faithful normal conditional expectations of a factor onto its subfactor, (both of which can be of type I, II, or III) with the same index greater than or equal to 9 (see 3.12). The values of indices in our example can not be less than 4, for, if $\mathrm{IND}_E = 4\cos^2 \pi/n$ for $n \ge 3$, by a theorem due to Haagerup [4], there is only one faithful normal conditional expectation of M onto its subfactor N. Recently, similar works on uncountable conjugacy classes of $*$-endomorphisms with the same index were done by Powers [8], Choda [2], and Bures–Yin [1]. One major tool used in deriving the above result is the formula for the index of the tensor product of two faithful normal conditional expectations. It states that for any E_i, faithful normal conditional expectations

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of a factor M_i onto its subfactor N_i, $i = 1, 2$, $\mathrm{IND}_{E_1 \otimes E_2} = (\mathrm{IND}_{E_1})(\mathrm{IND}_{E_2})$, where $E_1 \otimes E_2$ is a faithful normal conditional expectation of $M_1 \otimes M_2$ onto $N_1 \otimes N_2$ (see Theorem 3.11). In the case when M_i, N_i are factors of type II_1 and E_i are trace preserving, the result can be easily verified as in [5]. Using this theorem we can easily construct type III factors M, N and a faithful normal conditional expectation E of M onto N such that IND_E can be any number in $\{4\cos^2 \pi/n | n \geq 3\} \cup [4, \infty)$ (see 3.10).

2. Index of type I factors

Let M, N be type I factors and $N \subseteq M$. In this section we completely describe all faithful normal conditional expectations E from M onto N and their indices. Hence we may answer the following question completely. For E_1, E_2, normal faithful conditional expectations of M onto N, when does $\mathrm{IND}_{E_1} = \mathrm{IND}_{E_2}$ hold?

2.1. Let $M = B(\mathscr{H})$, the von Neumann algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$, and $N = \mathscr{C}$, where $\mathscr{H}$ is separable (may be of finite dimension) and $\mathscr{C}$ is the complex field. Any normal state φ on $B(\mathscr{H})$ is of the form $\varphi(x) = \mathrm{tr}(Ax)$ for some positive trace class operator A with $\mathrm{tr}(A) = 1$, where tr is the trace on $B(\mathscr{H})$. Let $\{\lambda_i\}$ be the set of all eigenvalues of A. If φ is faithful, then $\lambda_i > 0$ for all i. In this notation we have the following.

2.2. **Proposition.** $\mathrm{IND}_\varphi = \sum_{i=1}^{\infty} 1/\lambda_i$ *for any faithful normal state* φ *on* $B(\mathscr{H})$. *In particular* IND_φ *is finite if and only if* $\mathscr{H}$ *is of finite dimension.*

Proof. Let $N_\varphi = \{x \in M | \varphi(x^*x) < \infty\}$, $\mathscr{H}_\varphi$ be the Hilbert space completion of $\widehat{N}_\varphi$ with $\widehat{\ }$ denoting the embedding of N_φ into $\mathscr{H}_\varphi$ and π_φ the GNS representation induced by φ. As in [3], for each vector $\xi \in \mathscr{H}$, we define densely defined operator $R^\varphi(\xi)$ from $\mathscr{H}_\varphi$ to $\mathscr{H}$ by $R^\varphi(\xi)(\widehat{\ }x) = x\xi$, $\forall x \in N_\varphi$. If $\xi = A^{1/2}\zeta$ for some $\zeta \in \mathscr{H}$, with $\|\zeta\| = 1$, then $\|\widehat{\ }(x)\|^2 = \varphi(x^*x) = \mathrm{tr}(A^{1/2}x^x xA^{1/2}) \geq \|(xA^{1/2})^*(xA^{1/2})\| = \|xA^{1/2}\|^2 \geq \|xA^{1/2}\zeta\|^2 = \|x\xi\|^2$ and $R^\varphi(\xi)$ is bounded. For those ξ's we denote $R^\varphi(\xi)R^\varphi(\xi)^*$ by $\theta^\varphi(\xi, \xi)$. It is easily checked that $\theta^\varphi(\xi, \xi) \in M_+$. Let $\{\xi_i\}$ be the complete orthonormal set of eigenvectors of A corresponding to $\{\lambda_i\}$, and $\xi_i \odot \xi_j$ be the rank-one operation in $B(\mathscr{H})$ such that $(\xi_i \odot \xi_j)(\xi) = \langle \xi, \xi_i \rangle \xi_j$ for $\xi \in \mathscr{H}$. We claim that $\theta^\varphi(\xi_i, \xi_i) = I_{\mathscr{H}}/\lambda_i$. For, we have

$$R^\varphi(\xi_i)(\widehat{\ }(\xi_k \odot \xi_l)) = (\xi_k \odot \xi_l)(\xi_i) = \langle \xi_i, \xi_k \rangle \xi_l = \delta_{i,k}\xi_l,$$

and

$$\begin{aligned} \langle R^\varphi(\xi_i)^*(\xi_j), \widehat{\ }(\xi_k \odot \xi_l) \rangle &= \langle \xi_j, R^\varphi(\xi_i)(\widehat{\ }(\xi_k \odot \xi_l)) \rangle \\ &= \langle \xi_j, \delta_{i,k}\xi_l \rangle = \delta_{i,k}\delta_{j,l}. \end{aligned}$$

Since $\widehat{\ }(\xi_k \odot \xi_l)$ form an orthogonal basis for $\mathscr{H}_\varphi$, and

$$\begin{aligned} \|\widehat{\ }(\xi_i \odot \xi_j)\|^2 &= \varphi((\xi_i \odot \xi_j)^*(\xi_i \odot \xi_j)) \\ &= \mathrm{tr}(A(\xi_i \odot \xi_i)) = \langle A\xi_i, \xi_i \rangle = \lambda_i, \end{aligned}$$

we get $R^{\varphi}(\xi_i)^*(\xi_j) =\hat{}\ (\xi_i \odot \xi_j)/\lambda_i$ and hence $R^{\varphi}(\xi_i)R^{\varphi}(\xi_i)^*(\xi_j) = (1/\lambda_i)\xi_j$ $\forall i, j = 1, 2, \ldots$. Thus $\theta^{\varphi}(\xi_i, \xi_i) = I_{\mathscr{H}}/\lambda_i$, and

$$\begin{aligned}\mathrm{Ind}_{\varphi} = \varphi^{-1}(I) = \varphi^{-1}\left(\sum_{i=1}^{\infty}(\xi_i \odot \xi_i)\right) &= \sum_{i=1}^{\infty}\varphi^{-1}(\xi_i \odot \xi_i)\\ &= \sum_{i=1}^{\infty}\theta^{\varphi}(\xi_i, \xi_i)\\ &= \left(\sum_{i=1}^{\infty}1/\lambda_i\right)I_{\mathscr{H}}.\end{aligned}$$

The second to the last equality, which was first used in [6], can be easily verified. Furthermore since $\sum\lambda_i = 1$, $\lambda_i > 0$, we have $\mathrm{IND}_{\varphi} = \infty$ unless $\{\lambda_i\}$ is a finite set, or, equivalently $\mathscr{H}$ is of finite dimension n. In that case the minimum $\{\mathrm{IND}_{\varphi}|\varphi\} = n^2$. Q.E.D.

The next theorem is due to Murray and von Neumann as Theorem 8.6.1 in [7].

2.3. **Theorem.** *Let N be a factor of type I in $B(\mathscr{H})$ for a separable Hilbert space $\mathscr{H}$. Then $N \simeq B(\mathscr{H}_1)$ for a subspace $\mathscr{H}_1$ of $\mathscr{H}$, and $\mathscr{H} \cong \mathscr{H}_1 \otimes \mathscr{H}_2$ with $B(\mathscr{H}) \cong B(\mathscr{H}_1) \otimes B(\mathscr{H}_2)$, $N' \simeq B(\mathscr{H}_2)$.*

In the notation as in Theorem 2.3, we have:

2.4. **Proposition.** *The only normal conditional expectation E of $B(\mathscr{H})$ onto a subfactor N of type I is* $\mathrm{id} \otimes \varphi$, *where φ is a normal state on $B(\mathscr{H}_2)$, and* id *is the identity map on $B(\mathscr{H}_1)$.*

Proof. Denote a linear functional φ on $B(\mathscr{H}_2)$ by $\varphi(y) = E(I \otimes y)$ which is well defined. In fact, because $E(I \otimes y) \in B(\mathscr{H}_1) \otimes I$ and $E((x \otimes I)(I \otimes y)) = (x \otimes I)E(I \otimes y) = E((I \otimes y)(x \otimes I)) = E(I \otimes y)(x \otimes I)$, we have $E(I \otimes y) \in Z(B(\mathscr{H}_1) \otimes I)$, the center of $B(\mathscr{H}_1) \otimes I$, and hence $E(I \otimes y) = \mathrm{scalar} \cdot (I \otimes I)$. Then φ is a state, for $\varphi(I \otimes I) = (I \otimes I)$ and φ is positive. Then φ is also normal, for E is. Finally

$$\begin{aligned}E\left(\sum x_i \otimes y_i\right) &= \sum E((x_i \otimes I)(I \otimes y_i))\\ &= \sum (x_i \otimes I)\varphi(y_i) = \sum x_i \otimes \varphi(y_i)I\\ &= \sum(\mathrm{id} \otimes \varphi)(x_i \otimes y_i) = (\mathrm{id} \otimes \varphi)\sum x_i \otimes y_i,\\ &\qquad \forall x_i \in B(\mathscr{H}_1),\ y_i \in B(\mathscr{H}_2).\end{aligned}$$ Q.E.D.

In the same notation as in Theorem 2.3, let E be a faithful normal conditional expectation of $B(\mathscr{H})$ onto $B(\mathscr{H}_1)$ and $E = \mathrm{id} \otimes \varphi$. Identifying $B(\mathscr{H}_1)'$

with $B(\mathscr{H}_2)$, we show

2.5. **Theorem.** $E^{-1} = \varphi^{-1}$.

Proof. Let ψ be an arbitrary faithful normal state on N and $\imath$ be the identity state on $\mathscr{C}$. As in [6], E^{-1}, an operator valued weight from N' to M', is characterized by $\Delta(\psi \circ E|\imath) = \Delta(\psi|\imath \circ E^{-1})$, where Δ denotes the spatial derivative defined by Connes in [3]. Since the above equation completely determines E^{-1} for given E, we need only to show the following $\Delta(\psi \circ E|\imath) = \Delta(\psi|\imath \circ \varphi^{-1})$. In fact, for $\eta \in \mathscr{H}$,

$$\begin{aligned}\|[\Delta(\psi|\imath \circ \varphi^{-1})]^{1/2}(\eta)\|^2 &= \psi(\theta^{\varphi^{-1}}(\eta, \eta)) = \psi(R_\eta^{\varphi^{-1}} R_\eta^{\varphi^{-1}} *) \\ &= \psi(\varphi(\eta \otimes \eta)) = (\psi \otimes \varphi)(\eta \otimes \eta) \\ &= (\psi \otimes \varphi)(R_\eta^{\imath} R_\eta^{\imath *}) \\ &= \|[\Delta(\psi \circ E|\imath)]^{1/2}(\eta)\|^2. \quad \text{Q.E.D.}\end{aligned}$$

2.6. **Corollary.** $E^{-1} = \sum_{i=1}^{\infty} 1/\lambda_i$, *where* λ_i *are eigenvalue of* A *with* $\varphi(x) = \mathrm{tr}(Ax)$ *for all* $x \in N'$.

Proof. It follows straightforwardly from Theorem 2.5 and Proposition 2.2. Q.E.D.

2.7. **Corollary.** *Let* E *be a nontrivial faithful normal conditional expectation of a type* I *factor* M *onto its subfactor* N. *Then* $\mathrm{IND}_E \geq 4$.

Proof. Due to a theorem by Tomiyama [11], N has to be of type I. It follows from Corollary 2.6 that $\mathrm{IND}_E \geq 4$. Q.E.D.

2.8. Let E_1, E_2 be two faithful normal conditional expectations of a type I factor M onto its subfactor N. Then $\mathrm{IND}_{E_1} = \mathrm{IND}_{E_2}$ if and only if $\sum 1/\lambda_i = \sum 1/\gamma_i$, where $\{\lambda_i\}$ is the set of eigenvalues associated to E_1 and $\{\gamma_i\}$ is the set of eigenvalues associated to E_2 in Corollary 2.6. If there exists an automorphism θ of M, (which is actually inner, i.e., $\theta(x) = Ad_u(x)$ for some unitary u in M) such that $E_1 = E_2 \circ \theta$. Then $\{\lambda_i\} = \{\gamma_i\}$. Thus we can have uncountable family of nonconjugate faithful normal conditional expectations of type I factor M with the same index greater than 9. In fact, one can easily construct such a family of E's on $M = M_3$ and $N = \mathscr{C}$, where M_3 is the factor of all 3×3 complex matrices, as follows.

Let

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

with $\lambda_i > 0$, $i = 1, 2, 3$, and $\sum \lambda_i = 1$, $\sum 1/\lambda_i = \alpha > 9$. A faithful normal state φ_A on M_3 is defined by $\varphi_A(x) = \mathrm{tr}(Ax)$. From Proposition 2.2 we have $\mathrm{IND}_{\varphi_A} = 1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 = \alpha$. For each given $\alpha > 9$, $\{(\lambda_1, \lambda_2) | 0 < \lambda_i < 1$, $i = 1, 2, 0 < \lambda_1 \leq \lambda_2 \leq 1 - \lambda_1 - \lambda_2 < 1$ & $1/\lambda_1 + 1/\lambda_2 + 1/1 - \lambda_1 - \lambda_2 = \alpha\}$ is uncountable. Each such triple gives rise to a faithful normal state and these states are not pairwise unitarily equivalent. However, in the case of 2×2 matrix algebra, for any $\alpha \geq 4$ there can be at most two pairs of (λ_1, λ_2) such that $0 < \lambda_i < 1$, $i = 1, 2$, $\lambda_1 + \lambda_2 = 1$ and $1/\lambda_1 + 1/\lambda_2 = \alpha$.

3. Index of tensor products

3.1. Let M be a factor of type s and N be a subfactor of type t. If there exists a normal conditional expectation of M onto N then $s \geq t$ by a theorem of Tomiyama [11].

Suppose that there is a faithful normal conditional expectation E from M onto N with $\mathrm{IND}_E = 4\cos^2 \pi/n$ for $n > 3$. Then the relative commutant of N in M, denoted by N^C is $\mathscr{C}$ [6, 5]. By a theorem due to Haagerup in [4], E is the only faithful normal conditional expectation of M onto N.

3.2. Let E be a faithful normal conditional expectation of a factor M onto its subfactor N, and let R be a factor. We define a faithful normal conditional expectation $\widehat{E}$ of $M \otimes R$ onto $N \otimes R$ by $\widehat{E}(x \otimes r) = E(x) \otimes r$ for all $x \in M$, $r \in R$. Thus we show

3.3. **Theorem.** $\widehat{E}^{-1}(I) = E^{-1}(I)$.

The proof of the above theorem is based on the following propositions and lemmas.

Let φ be a faithful normal semifinite weight on $B(\mathscr{H})$ (the same notation as in §1). Then $\varphi(x) = \mathrm{tr}(Ax)$ for all x in $B(\mathscr{H})$, where A is a positive definite densely-defined closed operator on $\mathscr{H}$. Let $\{\lambda_i\}$ be the set of all positive eigenvalues of A and $\{\xi_i\}$ be corresponding orthonormal set of eigenvectors. Let ι be the identity functional on $\mathscr{C}$.

3.4. **Lemma.** *$\Delta(\varphi|\iota) = A$ and $\Delta(\varphi^{-1}|\iota) = A^{-1}$.*

Proof. For $\xi \in \mathscr{H}$, $\|\Delta(\varphi|\iota)^{1/2}\xi\|^2 = \varphi(\theta^\iota(\xi, \xi)) = \varphi(\xi \odot \xi) = \mathrm{tr}(A(\xi \odot \xi)) = \sum \lambda_i |\langle \xi, \xi_i\rangle|^2$. Thus $\Delta(\varphi|\iota)^{1/2} = A^{1/2}$ and $\Delta(\varphi|\iota) = A$. By [3, Theorem 9, condition (3)] and [6, 1.2] we have

$$\Delta(\varphi^{-1}|\iota) = \Delta(\iota|\varphi^{-1})^{-1} = \Delta(\varphi|\iota)^{-1} = A^{-1}. \quad \text{Q.E.D.}$$

3.5. **Proposition.** *Let φ_i be faithful normal semifinite weights on $B(\mathscr{H}_i)$ for $i = 1, 2$. Then $\varphi_1 \otimes \varphi_2$ is a faithful normal semifinite weight on $B(\mathscr{H}_1) \otimes B(\mathscr{H}_2) \simeq B(\mathscr{H}_1 \otimes H_2)$ and $(\varphi_1 \otimes \varphi_2)^{-1} = \varphi_1^{-1} \otimes \varphi_2^{-1}$.*

Proof. It is obvious that $\varphi_1 \otimes \varphi_2$ is a faithful normal semifinite weight on $B(\mathscr{H}_1) \otimes B(\mathscr{H}_2)$, which is isomorphic to $B(\mathscr{H}_1 \otimes \mathscr{H}_2)$. Let $\varphi_i(x) = \mathrm{tr}(A_i x)$ for $x \in B(\mathscr{H}_i)$, where A_i is a positive definite densely-defined closed operator in $B(\mathscr{H}_i)$, $i = 1, 2$. It is easy to see that $(\varphi_1 \otimes \varphi_2)(x) = \mathrm{tr}((A_1 \otimes A_2)x)$ for x in $B(\mathscr{H}_1 \otimes H_2)$ and $\mathrm{tr}(x_1 \otimes x_2) = \mathrm{tr}(x_1)\mathrm{tr}(x_2)$ for x_i in $B(\mathscr{H}_i)$, $i = 1, 2$. Then, it follows from Lemma 3.4 that $\Delta((\varphi_1 \otimes \varphi_2)^{-1}|\iota) = (A_1 \otimes A_2)^{-1}$, which in turn is equal to $A_1^{-1} \otimes A_2^{-1}$ and $\Delta(\varphi_i^{-1}|\iota) = A_i^{-1}$, $i = 1, 2$. Since $\Delta(\varphi|\iota)$ determines φ completely by Lemma 3.4, $(\varphi_1 \otimes \varphi_2)^{-1} = \varphi_1^{-1} \otimes \varphi_2^{-1}$. Q.E.D.

Let M_i be factors operating on $\mathscr{H}_i$, and φ_i, φ_i' be faithful normal semifinite weights on M_i, M_i' respectively, $i = 1, 2$. Then we show

3.6. **Proposition.** $(\varphi_1 \otimes \varphi_2)^{-1} = \varphi_1^{-1} \otimes \varphi_2^{-1}$.

Proof. Let $\imath$ be the identity state on $\mathscr{C}$. By the commutation theorem of tensor product we know $M_1' \otimes M_2' = (M \otimes M_2)'$ (see [9]). Since $\Delta(\imath(\varphi_1 \otimes \varphi_2)|\varphi_1' \otimes \varphi_2') = \Delta(\imath|(\varphi_1' \otimes \varphi_2')(\varphi_1 \otimes \varphi_2)^{-1})$, it suffices to show

$$\Delta(\imath(\varphi_1 \otimes \varphi_2)|\varphi_1' \otimes \varphi_2') = \Delta(\imath|(\varphi_1' \otimes \varphi_2')(\varphi_1^{-1} \otimes \varphi_2^{-1})).$$

For arbitrary vector $\xi_i \in \mathscr{H}_i$, consider

$$\begin{aligned} &\|\Delta(\imath|\varphi_1'\varphi_1^{-1} \otimes \varphi_2'\varphi_2^{-1})^{1/2}(\xi_1 \otimes \xi_2)\|^2 \\ &= \imath(\theta^{\varphi_1'\varphi_1^{-1}\otimes\varphi_2'\varphi_2^{-1}}((\xi_1 \otimes \xi_2), (\xi_1 \otimes \xi_2))) \\ &= \imath(\varphi_1'\varphi_1^{-1} \otimes \varphi_2'\varphi_2^{-1})^{-1}((\xi_1 \otimes \xi_2) \odot (\xi_1 \otimes \xi_2))). \end{aligned} \tag{3.6.1}$$

It follows from Proposition 3.5 that

$$(\varphi_1'\varphi_1^{-1} \otimes \varphi_2'\varphi_2^{-1})^{-1} = (\varphi_1'\varphi_1^{-1})^{-1} \otimes (\varphi_2'\varphi_2^{-1})^{-1} = \varphi_1\varphi_1'^{-1} \otimes \varphi_2\varphi_2'^{-1}$$

(the last equality is due to Haagerup in [4]). Thus, 3.6.1 becomes

$$\begin{aligned} &\imath(\varphi_1\varphi_1'^{-1} \otimes \varphi_2\varphi_2'^{-1}((\xi_1 \otimes \xi_2) \odot (\xi_1 \otimes \xi_2))) \\ &= \imath(\varphi_1 \otimes \varphi_2)(\varphi_1'^{-1} \otimes \varphi_2'^{-1})((\xi_1 \otimes \xi_2) \odot (\xi_1 \otimes \xi_2)) \\ &= \imath(\varphi_1 \otimes \varphi_2)(\varphi_1' \otimes \varphi_2')^{-1}((\xi_1 \otimes \xi_2) \odot (\xi_1 \otimes \xi_2)), \quad \text{by Proposition 3.5}, \\ &= \imath(\varphi_1 \otimes \varphi_2)(\theta^{\varphi_1'\otimes\varphi_2'}((\xi_1 \otimes \xi_2), (\xi_1 \otimes \xi_2))) \\ &= \|\Delta(\imath(\varphi_1 \otimes \varphi_2)|\varphi_1' \otimes \varphi_2')^{1/2}(\xi_1 \otimes \xi_2)\|^2. \end{aligned}$$

Hence $\Delta(\imath(\varphi_1 \otimes \varphi_2|\varphi_1' \otimes \varphi_2') = \Delta(\imath|(\varphi_1' \otimes \varphi_2')(\varphi_1^{-1} \otimes \varphi_2^{-1}))$. Q.E.D.

3.7. **Lemma.** $(E \otimes \mathrm{id})^{-1} = E^{-1} \otimes \mathrm{id}$.

Proof. Coming back to the setup in 3.2, we let φ, φ_r be faithful normal states on M', R' respectively, and ψ, ψ_r be faithful normal states on N, R respectively. Since $\Delta((\psi \otimes \psi_r)(E \otimes \mathrm{id})|\varphi \otimes \varphi_r) = \Delta(\psi \otimes \psi_r|(\varphi \otimes \varphi_r)(E \otimes \mathrm{id})^{-1})$ and the equation determines $(E \otimes \mathrm{id})^{-1}$ completely from $E \otimes \mathrm{id}$, it suffices to show $\Delta((\psi \otimes \psi_r)(E \otimes \mathrm{id})|\varphi \otimes \varphi_r) = \Delta(\psi \otimes \psi_r|(\varphi \otimes \varphi_r)(E^{-1} \otimes \mathrm{id}))$. For any vectors $\eta_i \in \mathscr{H}_i$, $i = 1, 2$, where M, $N \subseteq B(\mathscr{H}_1)$ and $R \subseteq B(\mathscr{H}_2)$, consider

$$\begin{aligned} &\|\Delta((\psi \otimes \psi_r)(E \otimes \mathrm{id})|\varphi \otimes \varphi_r)^{1/2}(\eta_1 \otimes \eta_2)\|^2 \\ &= (\psi \circ E \otimes \psi_r)(\theta^{\varphi\otimes\varphi_r}(\eta_1 \otimes \eta_2, \eta_1 \otimes \eta_2)) \\ &= (\psi \circ E \otimes \psi_r)((\varphi \otimes \psi_r)^{-1}((\eta_1 \otimes \eta_2) \odot (\eta_1 \otimes \eta_2))). \end{aligned} \tag{3.7.1}$$

It follows from Proposition 3.6 that $(\varphi \otimes \varphi_r)^{-1} = \varphi^{-1} \otimes \varphi_r^{-1}$ and 3.7.1 becomes

$$\begin{aligned}
&(\psi \circ E \otimes \psi_r)((\varphi^{-1} \otimes \varphi_r^{-1})((\eta_1 \otimes \eta_2) \odot (\eta_1 \otimes \eta_2))) \\
&\quad = (\psi \otimes \psi_r)(E \circ \varphi^{-1} \otimes \varphi_r^{-1})((\eta_1 \otimes \eta_2) \odot (\eta_1 \otimes \eta_2)) \\
&\quad = (\psi \otimes \psi_r)((\varphi \circ E^{-1})^{-1} \otimes \varphi_r^{-1})((\eta_1 \otimes \eta_2) \odot (\eta_1 \otimes \eta_2)) \\
&\quad = (\psi \otimes \psi_r)(\varphi \circ E^{-1} \otimes \varphi_r)^{-1}((\eta_1 \otimes \eta_2) \odot (\eta_1 \otimes \eta_2)), \quad \text{by Proposition 3.6}, \\
&\quad = (\psi \otimes \psi_r)(\theta^{\varphi \circ E^{-1} \otimes \varphi_r}(\eta_1 \otimes \eta_2, \eta_1 \otimes \eta_2)) \\
&\quad = \|\Delta(\psi \otimes \psi_r | (\varphi \otimes \varphi_r)(E^{-1} \otimes \mathrm{id}))^{1/2}(\eta_1 \otimes \eta_2)\|^2 .
\end{aligned}$$

Hence $\Delta((\psi \otimes \psi_r)(E \otimes \mathrm{id}) | \varphi \otimes \varphi_r) = \Delta(\psi \otimes \psi_r | (\varphi \otimes \varphi_r)(E^{-1} \otimes \mathrm{id}))$. Q.E.D.

3.8. Now, it is obvious that Theorem 3.3 follows easily from Lemma 3.7.

3.9. Let M be the hyperfinite II_1 factor and N its subfactor with the index $[M, N] \in \{4\cos^2 \pi/n | n \geq 3\} \cup [4, \infty)$, where E is the trace preserving faithful normal conditional expectation of M onto N. Let R be a type III_λ factor, then we exhibit a pair of type III_λ factors $M \otimes R$ and $N \otimes R$, $0 \leq \lambda \leq 1$, and a faithful normal conditional expectation $\widehat{E} = E \otimes \mathrm{id}$ such that $\mathrm{IND}_{\widehat{E}} = \mathrm{IND}_E = \{4\cos^2 \pi/n | n \geq 3\} \cup [4, \infty)$. In the case $[M, N] = \mathrm{IND}_E = 4\cos^2 \pi/n$ for $n \geq 3$, then E is the only faithful normal conditional expectation of M onto N.

3.10. **Theorem.** Let E_i be a faithful normal conditional expectation from a factor M_i onto its subfactor N_i, $i = 1, 2$. Then $\mathrm{IND}_{E_1 \otimes E_2} = (\mathrm{IND}_{E_1})(\mathrm{IND}_{E_2})$, where $E_1 \otimes E_2$ is a faithful normal conditional expectation of $M_1 \otimes M_2$ into $N_1 \otimes N_2$.

Proof. Denote $E_1 \otimes \mathrm{id}$, ($\mathrm{id} \otimes E_2$ respectively) by $\widehat{E}_1$, ($\widehat{E}_2$ respectively), and then $E_1 \otimes E_2 = \widehat{E}_1 \circ \widehat{E}_2$ and $(E_1 \otimes E_2)^{-1} = \widehat{E}_2^{-1} \circ \widehat{E}_1^{-1}$. By Theorem 3.3, we have $(E_1 \otimes E_2)^{-1}(I) = \widehat{E}_2^{-1}(\widehat{E}_1^{-1}(I)) = \widehat{E}_2^{-1}(E_1^{-1}(I)I) = E_1^{-1}(I)\widehat{E}_2^{-1}(I) = E_1^{-1}(I)E_2^{-1}(I)$. In fact, $\widehat{E}_2^{-1}(I) = E_2^{-1}(I)$ follows from a proof symmetric to that of Theorem 3.3. Q.E.D.

3.11. Let E_α be an uncountable family of nonconjugate faithful normal conditional expectations of a type I factor M onto its subfactor N considered in 2.8. For any factor R of type s, $s = \mathrm{I}$ or II or III, $M \otimes R$ is of type s and $\{E_\alpha \otimes \mathrm{id}\}$ is an uncountable family of nonouterconjugate faithful normal conditional expectations of $M \otimes R$ onto $N \otimes R$ with the same index. Similar works on an uncountable family of conjugacy classes of shifts with the same index on a type II_1-factor have recently been discussed in [2, 8].

3.12. *Concluding remarks.* For any $4 \leq \lambda < 9$ it is still unknown to the author how to find a pair of II_1-factors $M \supset N$ such that there are uncountable nonisomorphic faithful normal conditional expectations E_α's with $\mathrm{IND}_{E_\alpha} = \lambda$.

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