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WEAKLY INFINITE-DIMENSIONAL PRODUCT SPACES

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ABSTRACT. It is shown that the product of a weakly infinite-dimensional compactum with a C-space is weakly infinite-dimensional. Some observations on the coincidence of weak infinite-dimensionality and property C are made. The question of when a weakly infinite-dimensional space has weakly infinite-dimensional product with all zero-dimensional spaces is investigated.

1. INTRODUCTION

By a space we mean a metric space, and by the dimension of a space we mean the Lebesgue covering dimension, for example as presented in [E]. In this sequel paper, we continue the line of investigation, initiated in [R2], into when the product of two weakly infinite-dimensional spaces is itself weakly infinite-dimensional. The reader is referred to that source for a more complete discussion of the history of this question and specifically for the definitions of *countable-dimensional*, *weakly infinite-dimensional*, and *strongly infinite-dimensional* spaces, as well as these of *property* C, C*-space*, and C*-refinement.*

While it is known that every countable-dimensional space, hence every finite-dimensional space, has property C and that every C-space is weakly infinite-dimensional, R. Pol has constructed a compact C-space which is not countable-dimensional [P1]. It remains unknown whether or not every weakly infinite-dimensional space must have property C.

Similarly, while it is also known that the product of two C-spaces can be strongly infinite-dimensional [EP], [P2], it is still unknown whether or not the product of two weakly infinite-dimensional compacta must always be weakly infinite-dimensional. On the other hand, the productivity of property C for compacta has been established [R2].

Theorem. *The product of two* C*-spaces, one of which is compact, is itself a* C*-space.*

In this paper we investigate the productivity of weak infinite-dimensionality for compacta, showing productivity for a large, perhaps the entire, class of

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weakly infinite-dimensional factors. Finally, some observations concerning the coincidence of weak infinite-dimensionality and property C are given, along with some remarks on when a weakly infinite-dimensional space has weakly infinite-dimensional product with all zero-dimensional spaces.

2. Results and proofs

In proving our main result, we will make use of the following characterization of weak infinite-dimensionality in terms of binary open covers:

Lemma 1. *A space X is weakly infinite-dimensional if and only if, for any sequence of binary open covers $\{\mathscr{U}_n : n \in \mathbb{N}\}$ of X of the form $\mathscr{U}_n = \{U_n^1, U_n^2\}$, there exists a precise pairwise disjoint open refinement $\mathscr{V}_n$ of each U_n—i.e.:*

1. *For each $n \in \mathbb{N}$, $\mathscr{V}_n = \{V_n^1, V_n^2\}$, with V_n^1 and V_n^2 open in X.*
2. *For each $n \in \mathbb{N}$, $V_n^1 \cap V_n^2 = \varnothing$.*
3. *For each $n \in \mathbb{N}$, $V_n^1 \subset U_n^1$ and $V_n^2 \subset U_n^2$ —so that the $\bigcup\{\mathscr{V}_n : n \in \mathbb{N}\}$ forms an open cover of X.*

As the proof is elementary, involving only complete normality, we omit it. For this characterization and further generalizations of the notion, we refer the reader to [R1, Chapter 3].

Theorem 1. *The product of a weakly infinite-dimensional compactum with a C-space is again weakly infinite-dimensional.*

Proof. Given a C-space X and a weakly infinite-dimensional compactum Y, we show that the product $X \times Y$ is weakly infinite-dimensional. Let a countable collection of binary open covers of $X \times Y$ be given. We rewrite this collection as a sequence of such countable collections

$$\{\{\mathscr{U}_{m,n} : n \in \mathbb{N}\} : m \in \mathbb{N}\},$$

where each binary open cover has the form

$$\mathscr{U}_{m,n} = \{U_{m,n}^1, U_{m,n}^2\}.$$

Fix $m \in \mathbb{N}$, let $x \in X$ be fixed but arbitrary, and let $\pi : X \times Y \to X$ denote the projection mapping. For each $n \in \mathbb{N}$ and $\alpha \in \{1, 2\}$, we set

$$U_{m,n}^\alpha(x) = U_{m,n}^\alpha \cap \pi^{-1}(x) \quad \text{and} \quad \mathscr{U}_{m,n}(x) = \{U_{m,n}^1(x), U_{m,n}^2(x)\}.$$

Thus, $\mathscr{U}_{m,n}(x)$ is a binary open cover of $\pi^{-1}(x)$ for each $n \in \mathbb{N}$. Since $\pi^{-1}(x)$ is homeomorphic to Y, we will not, when the context is clear, distinguish between $\pi^{-1}(x)$ as a subspace of $X \times Y$, and Y.

In particular, $\pi^{-1}(x)$ is weakly infinite-dimensional, so that, using Lemma 1, we can choose subsets $V_{m,n}^1(x)$ and $V_{m,n}^2(x)$ of $\pi^{-1}(x)$ for each $n \in \mathbb{N}$ with

1. $V_{m,n}^1(x)$ and $V_{m,n}^2(x)$ open in $\pi^{-1}(x)$

2. $V^1_{m,n}(x) \cap V^2_{m,n}(x) = \varnothing$
3. $V^1_{m,n}(x) \subset U^1_{m,n}(x)$ and $V^2_{m,n}(x) \subset U^2_{m,n}(x)$,

so that $\{V^\alpha_{m,n}(x) : \alpha = 1, 2, n \in \mathbb{N}\}$ is a cover of $\pi^{-1}(x)$. We then use the compactness of $\pi^{-1}(x)$ to extract a finite subcover

$$\{V^\alpha_{m,n}(x) : \alpha = 1, 2, n = 1, \ldots, r_m(x)\}$$

for some positive integer $r_m(x)$, and, using normality, we "shrink" the elements of the finite subcover so that

$$V^\alpha_{m,n}(x) \subset \overline{V^\alpha_{m,n}(x)} \subset U^\alpha_{m,n}(x)$$

for each $\alpha \in \{1, 2\}$ and $n \in \{1, \ldots, r_m(x)\}$.

Next, we use an idea of Dieudonné [D] to construct an open cover of X.

Claim. For each $n \in \{1, \ldots, r_m(x)\}$, there is an open neighborhood $W_{m,n}(x)$ of x in X so that, for any $x' \in W_{m,n}(x)$ and $\alpha \in \{1, 2\}$, the inclusions

$$V^\alpha_{m,n}(x) \subset \overline{V^\alpha_{m,n}(x)} \subset U^\alpha_{m,n}(x')$$

hold.

Indeed if not, then we could choose a sequence (x_k, y_k) in $X \times Y$ with $x_k \to x$ where, without loss of generality, for each $k \in \mathbb{N}$,

$$y_k \in \overline{V^1_{m,n}(x)} \quad \text{but} \quad y_k \notin U^1_{m,n}(x_k).$$

By the compactness of Y, passing to a convergent subsequence if necessary, we have

$$y_k \to y \in \overline{V^1_{m,n}(x)} \subset U^1_{m,n}(x),$$

so that

$$(x_k, y_k) \to (x, y) \in U^1_{m,n}.$$

But then, since $U^1_{m,n}$ is open in $X \times Y$, we see that, for all sufficiently large k,

$$(x_k, y_k) \in U^1_{m,n}, \quad \text{so that } y_k \in U^1_{m,n}(x_k),$$

which is a contradiction.

We construct such an open set $W_{m,n}(x)$ for each $n \in \{1, \ldots, r_m(x)\}$ and set

$$W_m(x) = \bigcap\{W_{m,n}(x) : n = 1, \ldots, r_m(x)\}.$$

Then, $W_m(x)$ is an open neighborhood of $x \in X$, so that

$$\{W_m(x) \times V^\alpha_{m,n}(x) : \alpha = 1, 2, n = 1, \ldots, r_m(x)\}$$

is an open cover of $\pi^{-1}(x)$ in $X \times Y$. We form the open cover

$$\mathscr{W}_m = \{W_m(x) : x \in X\}$$

of X by constructing such a neighborhood $W_m(x)$ for each $x \in X$.

In this manner, we construct such an open cover $\mathscr{W}_m$ of X for each $m \in \mathbb{N}$. Since X has property C, we can choose a C-refinement $\mathscr{O}_m$ of $\mathscr{W}_m$ for each $m \in \mathbb{N}$ so that the $\cup\{\mathscr{O}_m : m \in \mathbb{N}\}$ covers X. Since each $\mathscr{O}_m$ refines $\mathscr{W}_m$, we can choose a function $\phi_m : \mathscr{O}_m \to X$ for each $m \in \mathbb{N}$ so that if $O \in \mathscr{O}_m$ we have

$$O \subset W_m(\phi_m(O)).$$

Thus, if $n \in \{1, \ldots, r_m(\phi_m(O))\}$ for some $O \in \mathscr{O}_m$, then

$$O \subset W_m(\phi_m(O)) \subset W_{m,n}(\phi_m(O)),$$

so that for $\alpha \in \{1, 2\}$ we have

$$O \times V^{\alpha}_{m,n}(\phi_m(O)) \subset W_{m,n}(\phi_m(O)) \times V^{\alpha}_{m,n}(\phi_m(O)) \subset U^{\alpha}_{m,n}.$$

For each $m, n \in \mathbb{N}$ and $\alpha \in \{1, 2\}$, we define

$$C^{\alpha}_{m,n} = \bigcup\{O \times V^{\alpha}_{m,n}(\phi_m(O)) : n \in \{1, \ldots, r_m(\phi_m(O))\} \text{ for some } O \in \mathscr{O}_m\}$$

and set

$$\mathscr{C}_{m,n} = \{C^1_{m,n}, C^2_{m,n}\}.$$

If $(x, y) \in C^{\alpha}_{m,n}$, then there exists $O \in \mathscr{O}_m$ with $n \in \{1, \ldots, r_m(\phi_m(O))\}$ so that

$$(x, y) \in O \times V^{\alpha}_{m,n}(\phi_m(O)) \subset U^{\alpha}_{m,n}.$$

Therefore, $\mathscr{C}_{m,n}$ is a precise open refinement of $\mathscr{U}_{m,n}$. Furthermore, since the elements of $\mathscr{O}_m$ are pairwise disjoint, and since

$$V^1_{m,n}(\phi_m(O)) \cap V^2_{m,n}(\phi_m(O)) = \varnothing$$

for any $O \in \mathscr{O}_m$ with $n \in \{1, \ldots, r_m(\phi_m(O))\}$, we see that

$$C^1_{m,n} \cap C^2_{m,n} = \varnothing.$$

Finally, since $\bigcup\{\mathscr{O}_m : m \in \mathbb{N}\}$ covers X, given a point $(x, y) \in X \times Y$ we can find $m \in \mathbb{N}$ and $O \in \mathscr{O}_m$ so that $x \in O$. Since $\pi^{-1}(\phi_m(O))$ is covered by

$$\{V^{\alpha}_{m,n}(\phi_m(O)) : \alpha = 1, 2, n = 1, \ldots, r_m(\phi_m(O))\},$$

we can also find $\alpha \in \{1, 2\}$ and $n \in \{1, \ldots, r_m(\phi_m(O))\}$ so that

$$y \in V^{\alpha}_{m,n}(\phi_m(O)).$$

Therefore, we see that

$$(x, y) \in O \times V^{\alpha}_{m,n}(\phi_m(O)) \subset C^{\alpha}_{m,n},$$

so that $\bigcup\{\mathscr{C}_{m,n} : m, n \in \mathbb{N}\}$ forms an open cover of $X \times Y$. By Lemma 1, we conclude that $X \times Y$ is weakly infinite-dimensional. $\square$

We single out two special cases of separate interest. In the second corollary, which we were unable to find in the literature, I denotes the closed unit interval.

Corollary 1. *The product of R. Pol's uncountable-dimensional compact C-space with any weakly infinite-dimensional compactum is again weakly infinite-dimensional.*

Corollary 2. *X is a weakly infinite-dimensional compactum if and only if $X \times I$ is a weakly infinite-dimensional compactum.*

Question 1. If $X \times I$ is weakly infinite-dimensional, then must X have property C?

In the final part of this note, we consider the following question.

Question 2. What properties must an infinite-dimensional space possess to ensure that its product with every zero-dimensional, hence countable-dimensional, space is a C-space (is weakly infinite-dimensional)?

Necessarily, such a space must itself be a C-space (weakly infinite-dimensional), but it is also known that this is not a sufficient condition for productivity with zero-dimensional factors [P]. On the other hand, as the following example shows, while we have shown that compactness of a C-space (weakly infinite-dimensional) factor is a sufficient condition for such productivity, it is not a necessary condition.

Recall that R. Pol's compactum, when constructed as a subspace of the Hilbert cube, has the form $P = X \cup B_1 \cup B_2$, where X is a topologically complete, totally disconnected, strongly infinite-dimensional subspace of the Hilbert cube with countable-dimensional remainder $B_1 \cup B_2 = P/X$. So constructed, B_1 and B_2 are disjoint Bernstein sets; i.e., all compact subsets of B_1 and B_2 are countable [P2]. It is known that $B_1 \cup X$ and $B_2 \cup X$ are noncompact C-spaces [EP].

We will also need the following classical result.

Lemma 2 [E, 4.3.6]. *If $f : X \to Y$ is a closed mapping between spaces X and Y where $\dim f^{-1}(y) \leq 0$ for each $y \in Y$, then $\dim X \leq \dim Y$.*

Theorem 2. *Given B_i, where $i \in \{1, 2\}$, and X as above, the product of $B_i \cup X$ with any zero-dimensional space Z is a C-space and thus is weakly infinite-dimensional.*

Proof. The proof follows ideas of [EP] and [P2]. We assume, without loss of generality, that $i = 1$, set $X_1 = (B_1 \cup X)$, and let Z be any zero-dimensional space. We will show that $X_1 \times Z$ has property C as a subspace of $P \times Z$.

Given a sequence $\{\mathscr{U}_n : n \in \mathbb{N}\}$ of covers of $X_1 \times Z$ by open subsets of $P \times Z$, for each fixed $n \in \mathbb{N}$ we set

$$K_n = (P \times Z) \setminus \bigcup \{U : U \in \mathscr{U}_n\}.$$

Each K_n is a closed subset of $P \times Z$ contained entirely in $B_2 \times Z$. Since the restricted projection $\pi : K_n \to Z$ is a closed mapping with countable, hence at most zero-dimensional, fibers, we apply Lemma 2 to see that $\dim K_n \leq 0$.

Being a countable union of closed zero-dimensional sets,

$$\dim \bigcup \{K_n : n \in \mathbb{N}\} \leq 0.$$

Thus, we can choose an at most zero-dimensional G_δ-subset $A \subset P \times Z$ [E, 4.1.19] so that

$$\bigcup \{K_n : n \in \mathbb{N}\} \subset A.$$

Then, we also have

$$\begin{aligned}(P \times Z) \setminus A &\subset (P \times Z) \setminus \bigcup \{K_n : n \in \mathbb{N}\} \\ &\subset \bigcap \{(P \times Z) \setminus K_n : n \in \mathbb{N}\} \\ &\subset \bigcap \{\cup\{U : U \in \mathscr{U}_n\} : n \in \mathbb{N}\},\end{aligned}$$

and, in particular, for each $n \in \mathbb{N}$ we see that $\mathscr{U}_n$ is an open cover of $(P \times Z) \setminus A$ in $P \times Z$. Since P is a compact C-space, $P \times Z$ is a C-space, and, being an F_σ subset of $P \times Z$, $(P \times Z) \setminus A$ is also a C-space [AG]. Thus, we can choose a C-refinement $\mathscr{V}_n$ of $\mathscr{U}_n$ for each $n > 1$ so that the $\bigcup \{\mathscr{V}_n : n > 1\}$ is a cover of $(P \times Z) \setminus A$, hence also a cover of $(X_1 \times Z) \setminus A$.

Finally, since $(X_1 \times Z) \cap A \subset A$, we see that $(X_1 \times Z) \cap A$ can be at most zero-dimensional. Therefore, we can choose a C-refinement $\mathscr{V}_1$ of the remaining cover $\mathscr{U}_1$ so that $\mathscr{V}_1$ still covers $(X_1 \times Z) \cap A$. Then, the $\bigcup \{\mathscr{V}_n : n \in \mathbb{N}\}$ is now a cover of all of $X_1 \times Z$, which completes the proof that $X_1 \times Z$ has property C. $\square$

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