PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 3, March 1991

FREE LOOP SPACES OF FINITE COMPLEXES HAVE INFINITE CATEGORY

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(Communicated by Frederick R. Cohen)

ABSTRACT. Let X be a 1-connected space such that each $H_i(X; \mathbb{Z})$ is finitely generated. In this paper we prove that if the reduced homology of X with coefficients in a field is nonzero, then the Lusternik-Schnirelmann category of the free loop space is infinite.

The study of the homology of free loop spaces is motivated by its various applications—for instance, to problems about the existence of closed geodesics on Riemannian manifolds and to problems in string theory, elliptic cohomology, and cyclic homology.

By definition, the free loop space ΛX of a simply connected space X is the space of all continuous maps from the circle into X. If $\Bbbk$ is any commutative field, we are interested in the homology $H_*(\Lambda X; \Bbbk)$. Most of the work on this subject has been done over characteristic zero fields, using the theory of Sullivan minimal models [12–15].

As McCleary says in a recent survey [9], "generalizing the results of rational homotopy to obtain mod r results is something of a Holy Grail." The purpose of this note is to answer one of McCleary's questions in [9].

To be more precise, we prove the following theorem:

Main Theorem. *Let X be a 1-connected space such that each $H_i(X; \mathbb{Z})$ is finitely generated, and let $\Bbbk$ be a field. Suppose that the category of X is finite and there exists an integer $r > 0$, such that $H^r(X; \Bbbk) \neq 0$. Then the category of the free loop space ΛX is infinite, and*

$$\dim H^*(\Lambda X; \Bbbk) = \infty.$$

Let us recall the definition of the category of a space S: $\operatorname{cat} S$ is an integer satisfying: $\operatorname{cat} S \leq n$ iff S can be covered by $n+1$ open sets, each of which is contractible in S. If there does not exist such an integer n, we say that $\operatorname{cat} S = \infty$.

Received by the editors September 14, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55P50, 55P62.
This research is partially supported by a NATO grant.

If $\Bbbk$ is a field of characteristic zero, then the fact that $\dim H^*(\Lambda X; \Bbbk) = \infty$ is a first result proved by Sullivan in 1973, using his minimal models [13]. Thus our main theorem is a direct consequence of the mapping theorem ([3, Theorem 5.1.] applied to the fibration $\Omega X \xrightarrow{j} \Lambda X \xrightarrow{p} X$, where p is the map sending a loop on its origin. Indeed the mapping theorem goes as follows: If a continuous map $f\colon X \to Y$ between 1-connected spaces induces an injection on rational homotopy groups, then

$$\mathrm{cat}_0(X) \leq \mathrm{cat}_0(Y).$$

Here $\mathrm{cat}_0(\cdot)$ is the category of the $\mathbb{Q}$-localization. As the free loop space fibration admits a section, $\pi_*(j)\colon \pi_*(\Omega X) \to \pi_*(\Lambda X)$ is injective, and so the mapping theorem gives:

$$\mathrm{cat}(\Lambda X) \geq \mathrm{cat}_0(\Lambda X) \geq \mathrm{cat}_0(\Omega X) = \infty.$$

Another proof, when $\Bbbk = \mathbb{Q}$, is given by Fadell and Husseini [2].

In view of these remarks, we have only to prove the theorem when $\Bbbk = \mathbb{Z}/p\mathbb{Z}$ and $H^+(X; \mathbb{Q}) = 0$. For that, we shall make use of the theory of elliptic Hopf algebras as developed in [4, 5, 6, and 8].

We deal with connected co-commutative graded Hopf algebras G of finite type over a field $\Bbbk$. The dual of G, G^v, is then a commutative graded Hopf algebra. For simplicity, we refer to G and G^v simply as a "Hopf algebra" and a "dual Hopf algebra." Recall first that for any connected, graded $\Bbbk$-algebra A,

$$\operatorname{depth} A = \inf\{n \mid \operatorname{Ext}^n_A(\Bbbk, A) \neq 0\},$$

and A is Gorenstein if and only if $\dim_{\Bbbk} \operatorname{Ext}_A(\Bbbk, A) = 1$. We say that A has *polynomial growth* if for some integer $r \geq 0$, and a constant $C > 0$,

$$\dim A_n \leq C.n^r, \text{ for all } n.$$

Now a Hopf algebra G is *elliptic* if one of the following equivalent conditions [8; Theorem C] is satisfied:

(1) G is a nilpotent Hopf algebra that is finitely generated as an algebra.
(2) G has finite depth and polynomial growth.

According to [4, Theorem 4.2], every elliptic Hopf algebra is Gorenstein. The study of such Hopf algebras is motivated by the following two theorems, which play a key role in our proof of the main theorem above:

Theorem A [4]. *Let X be a 1-connected space and $\Bbbk$ be a field. If $H_n(X; \Bbbk)$ is finite-dimensional for every n, then*

$$\operatorname{depth} H_*(\Omega X; \Bbbk) \leq \mathrm{cat}(X).$$

Theorem B [5]. *Let X be a 1-connected space with finite category and $\Bbbk$ be a field. If $H_n(X; \Bbbk)$ is finite-dimensional for every n and if $H_*(\Omega X; \Bbbk)$ is an elliptic Hopf algebra, then $H^*(X; \Bbbk)$ is finite-dimensional and satisfies Poincaré duality.*

In §1, we study the behavior of the depth of the homologies in a covering. This will be used in §2, which is completely devoted to the proof of our theorem.

1. Multiplicative fibrations of H-spaces

The next two propositions will permit the reduction in §2 to the case in which the space X is 2-connected.

Recall that a Hopf space is a space together with a continuous homotopy associative multiplication with a homotopy unit.

Proposition 1.1. *Suppose $Y \xrightarrow{i} X \xrightarrow{p} K(\mathbb{Z}, 2)$ is a multiplicative fibration of path-connected Hopf spaces, whose homologies with coefficients in a perfect field $\mathscr{k}$ have finite type. Then,*

$$\operatorname{depth} H_*(Y; \mathscr{k}) \leq \operatorname{depth} H_*(X; \mathscr{k}).$$

Proof of 1.1. Consider $p_*: H_*(X; \mathscr{k}) \to H_*(K(\mathbb{Z}, 2); \mathscr{k})$. Its cokernel K (which exists because $H_*(K(\mathbb{Z}, 2); \mathscr{k})$ is commutative) is dual to the Hopf kernel of p^*, $\ker p^*$. Since $H^*(K(\mathbb{Z}, 2); \mathscr{k}) = \mathscr{k}[x]$, any sub-Hopf algebra (and in particular $\ker p^*$) is either $\mathscr{k}$ or a polynomial algebra in one variable. Thus $\operatorname{Tor}^{\operatorname{Ker} p^*}(\mathscr{k}, \mathscr{k})$ is either $\mathscr{k}$ or else an exterior algebra in one variable of homological degree 1. Dually, $\operatorname{Cotor}^K(\mathscr{k}, \mathscr{k})$ is either $\mathscr{k}$ or else an exterior Hopf algebra in one variable u of homological degree -1.

Now by [11, Proposition 5.1.] the Eilenberg–Moore spectral sequence for the fibration converges from

$$E^2 = \ker p_* \otimes \operatorname{Cotor}^K(\mathscr{k}, \mathscr{k}), \qquad E^2_{0,*} = \ker p_*,$$

to $H_*(Y)$, where again $\ker p_*$ denotes the Hopf kernel. Since $\operatorname{Cotor}^K(\mathscr{k}, \mathscr{k}) = \mathscr{k}$ or Λu, we have $E^2 = E^\infty$, and the equality $E^\infty_{0,*} = E^2_{0,*}$ identifies $\operatorname{im} i_* = \ker p_*$.

If $\operatorname{Cotor}^K(\mathscr{k}, \mathscr{k}) = \mathscr{k}$, then $H_*(Y) \xrightarrow{\cong} \operatorname{Ker} p_*$. In the other case there are linear isomorphisms of graded vector spaces

$$\operatorname{im} i_* \otimes \operatorname{Ker} i_* \cong H_*(Y) \cong E^\infty \cong \ker p_* \otimes \Lambda u$$

which imply that $\ker i_*$ also has the form Λv. In this case [4, Proposition 3.1] applied to the short exact sequence

$$\mathscr{k} \to \Lambda v \to H_*(Y) \to \ker p_* \to \mathscr{k}$$

gives depth $H_*(Y) = \operatorname{depth} \ker p_*$. Because $\ker p_*$ is normal, depth $H_*(Y) = \operatorname{depth} \ker p_* \leq \operatorname{depth} H_*(X)$.

Proposition 1.2. *Let $Y \to X \xrightarrow{p} K(\Gamma, 1)$ be a multiplicative fibration of path-connected Hopf spaces with Γ a finitely generated Abelian group and $\mathscr{k}$ a field. Then*

$$\operatorname{depth} H_*(Y; \mathscr{k}) \leq \operatorname{depth} H_*(X; \mathscr{k}).$$

Proof. It is clearly enough to prove the proposition when $\mathbb{k}$ is a perfect field. We consider a finitely generated $\mathbb{Z}$-free resolution of Γ:

$$0 \to L_1 \to L_0 \to \Gamma \to 0.$$

By looping the maps in the fiber sequence

$$K(L_0, 2) \to K(\Gamma, 2) \to K(L_1, 3),$$

we obtain a strictly multiplicative fibration

$$K(L_0, 1) \to K(\Gamma, 1) \xrightarrow{\beta} K(L_1, 2).$$

Let F be the fiber of the composite

$$q: X \xrightarrow{p} K(\Gamma, 1) \xrightarrow{\beta} K(L_1, 2).$$

The fibers Y, F, and $K(L_0, 1)$ of p, q, and β fit into the strictly multiplicative fibration of Hopf spaces

$$Y \to F \to K(L_0, 1).$$

As L_0 is $\mathbb{Z}$-free and Y is path-connected, F has the weak homotopy type of the product $Y \times K(L_0, 1)$. This implies the exactness of the sequence of Hopf algebras

$$\mathbb{k} \to H_*(Y; \mathbb{k}) \to H_*(F; \mathbb{k}) \to H_*(K(L_0, 1); \mathbb{k}) \to \mathbb{k}.$$

Since $K = H_*(K(L_0, 1); \mathbb{k})$ is finite-dimensional, by [7, (Theorem II)], we have depth $H_*(F; \mathbb{k}) =$ depth $H_*(Y; \mathbb{k}) +$ depth $K =$ depth $H_*(Y; \mathbb{k})$.

Finally, we may suppose $K(L_1, 2) = \Pi_{i=1}^{r} K(\mathbb{Z}, 2)$. Denote by F_n the homotopy fiber of the composite map

$$X \xrightarrow{q} K(L_1, 2) \xrightarrow{p_n} \Pi_{i=1}^{n} K(\mathbb{Z}, 2), \qquad 1 \le n \le r.$$

We then have a sequence of multiplicative fibrations

$$F_n \to F_{n-1} \to K(\mathbb{Z}, 2)$$

and so, by Proposition 1.1, we have the sequence of inequalities

$$\text{depth } H_*(Y; \mathbb{k}) = \text{depth } H_*(F_r; \mathbb{k}) \le \cdots \le \text{depth } H_*(X; \mathbb{k}).$$

2. Proof of the main theorem

The proof is based on two lemmas:

Lemma 1. *Let Y be a 2-connected space with $\mathbb{k}$-homology of finite type and such that* depth $H_*(\Omega Y; \mathbb{k})$ *and* depth $(\Omega\Lambda Y; \mathbb{k})$ *are finite. Then $H_*(\Omega Y; \mathbb{k})$ is an elliptic Hopf algebra.*

If X is a 1-connected space, denote by

$$X_{[2]} \xrightarrow{f} X \xrightarrow{s} K(\pi_2(X), 2)$$

the fibration in which $X_{[2]}$ is the 2-connected cover of X.

Lemma 2. *If* X *is a* 1*-connected space with finite type* $\mathbb{k}$*-homology and finite category, then* depth $H_*(\Omega X_{[2]}; \mathbb{k})$ *is finite.*

Given these lemmas, we achieve the proof of the main theorem as follows. We may assume, as mentioned in the introduction, that $H_+(X; \mathbb{Q}) = 0$, and so we need only to prove the theorem for $\mathbb{k} = \mathbb{Z}/p$. Now $H_+(\Lambda x; \mathbb{Q}) = 0$, and so each $H_i(\Lambda X; \mathbb{Z}) = 0$ is a finite Abelian group. Thus if $\dim H_*(\Lambda X; \mathbb{k})$ is finite, then localizing at p we find that $\Lambda(X_{(p)}) = (\Lambda X)_{(p)}$ has the homotopy type of a finite complex and hence finite category. Thus it is enough to show that the hypothesis that cat ΛX is finite leads to a contradiction. In the same way we may assume that each $H_i(X; \mathbb{Z})$ and $\pi_i(X)$ are finite Abelian p-groups.

It is easy to see that the category of the universal covering $\tilde{Y}$ of a space Y is always less than or equal to the category of Y, so that cat $\widetilde{\Lambda X}$ is finite. Consider now the composite

$$q\colon \widetilde{\Lambda X} \to \Lambda X \xrightarrow{p} X \xrightarrow{s} K(\pi_2(X), 2)$$

$\Lambda f\colon \Lambda X_{[2]} \to \Lambda X$ factors through $\tilde{\Lambda} X$, and we obtain a fibration

$$\Lambda X_{[2]} \to \widetilde{\Lambda X} \xrightarrow{q} K(\pi_2(X), 2).$$

Then, $H_*(\widetilde{\Lambda X}; \mathbb{k})$ has finite type, and so by [4, Theorem A] $H_*(\Omega\tilde{\Lambda} X; \mathbb{k})$ has finite depth. By taking the loops on this fibration, we obtain a multiplicative fibration of Hopf spaces

$$\Omega\Lambda X_{[2]} \to \Omega\widetilde{\Lambda X} \xrightarrow{\Omega_q} K(\pi_2(X), 1).$$

Proposition 1.2. then gives the inequality

$$\operatorname{depth} H_*(\Omega\Lambda X_{[2]}; \mathbb{k}) \le \operatorname{depth} H_*(\Omega\widetilde{\Lambda X}; \mathbb{k}) < \infty.$$

This, together with Lemma 2, shows that the space $X_{[2]}$ satisfies the hypothesis of Lemma 1. Therefore, the algebra $H_*(\Omega X_{[2]}; \mathbb{k})$ has polynomial growth. The cohomology Serre spectral sequence associated with the fibration

$$\Omega X_{[2]} \to \Omega X \to K(\pi_2(X), 1)$$

shows that $H_*(\Omega X; \mathbb{k})$ also has polynomial growth. Therefore, [Theorem B], $H_*(\Omega X; \mathbb{k})$ is an elliptic Hopf algebra and a Gorenstein algebra.

By [5, Proposition 3.2., IV and Theorem 3.1.], we deduce that $H^*(X; \mathbb{k})$ is finite-dimensional and satisfies Poincaré duality. Suppose $H^n(X; \mathbb{k}) \neq 0$ and $H^{>n}(X; \mathbb{k}) = 0$. This implies $H^n(X; \mathbb{Q}) \neq 0$, in contradiction to our assumption. □

It remains to prove Lemmas 1 and 2.

Proof of Lemma 2. The map s induces a multiplicative fibration of Hopf spaces

$$\Omega X_{[2]} \to \Omega X \to K(\pi_2(X), 1).$$

Proposition 1.2. and [4, Theorem A] show then that depth $H_*(\Omega X_{[2]}; \mathscr{k})$ is finite. □

Proof of Lemma 1. Since the fibration $\Omega Y \to \Lambda Y \xrightarrow{p} Y$ has a section, we have a short exact sequence of connected, graded Hopf algebras

$$\mathscr{k} \to H_*(\Omega^2 Y; \mathscr{k}_{\to} H_*(\Omega\Lambda Y : \mathscr{k}) \to H_*(\Omega Y; \mathscr{k}) \to \mathscr{k}.$$

The first term, $H_*(\Omega^2 Y; \mathscr{k})$ is Abelian and of finite depth as a normal sub-Hopf algebra of $H_*(\Omega\Lambda Y; \mathscr{k})$. Therefore, by [4, Theorem 4.2.], $H_*(\Omega^2 Y; \mathscr{k})$ is a nilpotent Gorenstein–Hopf algebra, and by [5, Proposition 3.2. (IV)], $\mathscr{C}_*(\Omega^2 Y; \mathscr{k})$ is a Gorenstein differential graded algebra. Moreover, from [5, Theorem 2.1.], we have the isomorphism

$$\mathrm{Ext}_{\mathscr{C}^*(\Omega Y)}(\mathscr{k}, \mathscr{C}^*(\Omega Y)) \cong \mathrm{Ext}_{\mathscr{C}_*(\Omega^2 Y)}(\mathscr{k}, \mathscr{C}_*(\Omega^2 Y)) \cong \mathscr{k},$$

so that $\mathscr{C}^*(\Omega Y; \mathscr{k})$ is also a Gorenstein differential graded algebra. The convergence of the Moore spectral sequence for differential Ext implies then that

$$\mathrm{Ext}_{H^*(\Omega Y; \mathscr{k})}(\mathscr{k}, H^*(\Omega Y; \mathscr{k})) \neq 0.$$

By Borel's theorem [1, Theorem 6.1.], $H^*(\Omega Y; \mathscr{k})$ is a tensor product of monogenic algebras. If there are infinitely many tensorands, then we could decompose $H^*(\Omega Y; \mathscr{k})$ in the form

$$H^*(\Omega Y; \mathscr{k}) = A_1 \otimes A_2 \otimes \cdots \otimes A_r,$$

with each A_i an infinite tensor product. Since each A_i is infinite-dimensional, it must have positive depth. Since the depth of a tensor product is equal to the sum of the depths of the factors, we would have

$$\text{depth } H^*(\Omega Y; \mathscr{k}) \geq r, \qquad \text{for every } r$$

which is impossible. Therefore, $H^*(\Omega Y; \mathscr{k})$ is the tensor product of finitely many monogenic algebras, and so has polynomial growth. Thus $H_*(\Omega Y; \mathscr{k})$ also has polynomial growth. Since it has finite depth by hypothesis, it is elliptic. □

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