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MEASURES ON BOOLEAN ALGEBRAS

WIESŁAW GŁÓWCZYŃSKI

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ABSTRACT. We give, under some set-theoretical assumptions, an example of complete, ccc, weakly (ω, ∞)-distributive, countably generated Boolean algebra without any strictly positive Maharam submeasure.

The problem of the existence of a complete, ccc, weakly (ω, ∞)-distributive Boolean algebra is an old one. Maharam [M] solved it assuming existence of a Suslin line; see also [V3] and [F].

We use the topological definitions as in [E], the Boolean measure definitions as in [F], and the set-theoretical ones as in [J]. We also use the standard notation, in particular $\forall_k^\infty$, MA, $\neg$ CH abbreviate "for all but finitely many k's," Martin's Axiom, and negation of the continuum hypothesis. The symbols $\wedge$, $\vee$, $\triangle$ denote infimum, supremum and symmetric difference in Boolean algebras; by $\mathscr{P}(\lambda)$ we mean the power set of λ.

For $f, g \in {}^\omega\omega$, we say that $g \leq^* f$ iff the set $\{n \in \omega | f(n) < g(n)\}$ is finite and let (see [D])

$$\underline{b} := \min\{|\mathscr{H}| | \mathscr{H} \subseteq {}^\omega\omega \text{ and } \neg(\exists f \in {}^\omega\omega)(\forall g \in \mathscr{H}) g \leq^* f\}.$$

We shall say that the sequence $\{x_n\}$ of subsets of cardinal λ converges to a subsets x of λ if and only if $\bigcap_{k\in\omega} \bigcup_{n\geq k}(x \triangle x_n) = \varnothing$ (in symbols, $x_n \to x$).

The following properties of $\to$ convergence in the Boolean algebra $\mathscr{P}(\lambda)$ are easy to verify:

(L0) If $x_n \to x$ and $x_n \to y$, then $x = y$.
(L1) If $x_n = x$ for all n, then $x_n \to x$.
(L2) If $x_n \to x$, then any subsequence also converges to x.
(L3) If $x_n \nrightarrow x$ (i.e., it is false that $x_n \to x$), then there is a subsequence y_m of x_n such that, for any subsequence z_p of y_m, $z_p \nrightarrow x$.

It means that the pair $(\mathscr{P}(\lambda), \to)$ is an L^* (see [E] for definitions of an L^* space and of an S^* space). But in some models of set theory, we have more.

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Proposition 1. *If* $\lambda < \underline{b}$, *then the pair* $(\mathscr{P}(\lambda), \to)$ *satisfies the following diagonal, Fréchet's condition*:

(L4) *If* $x_n \to x$ *for* $n \to \infty$ *and for each* n, $x_{n,k} \to x_n$ *for* $k \to \infty$, *then there exists a sequence of numbers* $g(n)$ *such that* $x_{n,g(n)} \to x$ *for* $n \to \infty$.

It means that the pair $(\mathscr{P}(\lambda), \to)$ *is an* S^* *space.*

Proof. Let $x_n \to x$. Then

$$x = \bigcap_k \bigcup_{n \geq k} x_n = \bigcup_k \bigcap_{n \geq k} x_n .$$

So for each $\alpha \in \lambda$ there exists m_α such that

$$(\forall_{n \geq m_\alpha} \alpha \in x_n) \quad \text{or} \quad (\forall_{n \geq m_\alpha} \alpha \notin x_n).$$

Because $x_{n,k} \to x_k$ so

$$(\forall_{n \geq m_\alpha} \forall_k^\infty \alpha \in x_{n,k}) \quad \text{or} \quad (\forall_{n \geq m_\alpha} \forall_k^\infty \alpha \notin x_{n,k}),$$

and hence

$$\forall_{n \geq m_\alpha} \exists_{g_\alpha(n)} (\forall_{k \geq g_\alpha(n)} \alpha \in x_{n,k}) \quad \text{or} \quad (\forall_{k \geq g_\alpha(n)} \alpha \notin x_{n,k}).$$

So for fixed α we may obtain a function $g_\alpha \colon \omega \to \omega$. If $\lambda < \underline{b}$, then there is $g \colon \omega \to \omega$ such that $g_\alpha \prec g$ for each $\alpha < \lambda$. It is easy to see that $x_{n,g(n)} \to x$. □

Into the set $\mathscr{P}(\lambda)$, we introduce the following topology τ: we call a set U *open* if whenever $x \in U$ and $x_n \to x$ then $x_n \in U$ for n sufficiently large.

Lemma 1. (i) $(\mathscr{P}(\lambda), \tau)$ *is a* T_1, *a sequential topological space and the* $\to$ *convergence is the same as topological convergence.*

(ii) *If* $\lambda < \underline{b}$, *then* $(\mathscr{P}(\lambda), \tau)$ *is a Fréchet space.*

Proof. For (i), see [Ki]; for (ii), [E, pp. 90–91] and Proposition 1. □

We give some other topological properties of $(\mathscr{P}(\lambda), \tau)$.

Proposition 2. *The space* $(\mathscr{P}(\lambda), \tau)$ *is*

(i) *homogeneous,*
(ii) *Hausdorff,*
(iii) *not regular for* $\lambda > \omega$, *and*
(iv) *sequentially compact for* $\lambda < 2^\omega$ *and under* MA.

Proof. For (i) and (ii), see [S].

(iii) Let C denote the set of all countable subsets of ω_1. The set C is closed. We show that for any open neighborhood U of $\omega_1 \in \mathscr{P}(\lambda)$ the intersection $C \cap \operatorname{cl} U \neq \varnothing$.

We use an Ulam matrix. Let $f_\alpha \colon \omega \to \alpha + 1$, for $\alpha < \omega_1$, be surjections. We define

$$A_{\alpha,n} := \{\xi \in \omega_1 | f_\xi(n) = \alpha\}.$$

The matrix $\{A_{\alpha,n}\}_{\alpha<\omega_1}^{n<\omega}$ has the following two properties: first, $\bigcup_{n\in\omega} A_{\alpha,n} = \{\xi|\alpha \leq \xi < \omega_1\}$; and second, if $\alpha < \beta < \omega_1$, then $A_{\alpha,n} \cap A_{\beta,n} = \varnothing$.

For each $\alpha < \omega_1$ we have:

$$\omega_1 = \alpha \cup \bigcup_{k\in\omega} \bigcup_{n\leq k} A_{\alpha,n} = \bigcup_{k\in\omega} \left(\alpha \cup \bigcup_{n\leq k} A_{\alpha,n} \right).$$

Let $B_{\alpha,k} := \alpha \cup \bigcup_{n\leq k} A_{\alpha,n}$. The set U is open in $\mathscr{P}(\lambda)$, so for each $\alpha < \omega_1$ there exists $k_\alpha < \omega$ such that $B_{\alpha,k_\alpha} \in U$. There are $k < \omega$ and $S \subseteq \omega_1$ of cardinality ω_1 such that for each $\alpha \in S$ we have $k_\alpha = k$.

Let $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$ be a sequence of elements of S and $\beta := \sup \alpha_n$; let $b_n := B_{\alpha_n,k}$. We claim that $b_n \to \beta$. The nonobvious inclusion is $\bigcap_k \bigcup_{n\geq k} b_n \subseteq \beta$. Suppose this inclusion is not true. Then there exists $\xi \geq \beta$ such that for infinitely many n we have $\xi \in \bigcup_{l\leq k} A_{\alpha_n,l}$ and there exists l_0 such that $\xi \in A_{\alpha_n,l_0}$ for infinitely many n. It is not possible because $A_{\alpha_n,l} \cap A_{\alpha_m,l} = \varnothing$ for $n \neq m$.

It follows that $\beta \in \operatorname{cl} U$. Since $\beta \in C$ as well, we obtain the result.

(iv) If MA holds then for $\lambda < 2^\omega$, the set $\mathscr{P}(\lambda)$ with Tychonoff topology (which is obviously weaker than τ) is a sequentially compact space; see [M-S]. □

Corollary 1. *The space* $(\mathscr{P}(\lambda), \tau)$ *with operation* $\triangle$ *is not a topological group for* $\lambda > \omega$.

Proof. It follows from Proposition 2(iii). □

Remark 1. Corollary 1 answers (without any set-theoretical assumptions) the question posed by Savelev [S].

Let $I \subseteq \mathscr{P}(\lambda)$ be a σ-ideal such that the quotient algebra $\mathscr{P}(\lambda)/I$ satisfies the countable chain condition (ccc for short). In the complete Boolean algebra $\mathscr{P}(\lambda)/I$, we induce a topology $\overline{\tau}$ by the following convergence:

$$x_n \Rightarrow x \text{ iff } \bigwedge_{k\in\omega} \bigvee_{n\geq k} (x \triangle x_n) = 0,$$

i.e., in the same way as τ by $\to$ on $\mathscr{P}(\lambda)$.

The $\Rightarrow$ convergence satisfies conditions (L0), (L1), (L2).

The relation between $\Rightarrow$ convergence and the topological convergence for the topology $\overline{\tau}$ is given by the following:

Lemma 2. *The topology* $\overline{\tau}$ *is the same as the topology induced by (topological) convergence in* $\overline{\tau}$.

The sequence x_n *topologically converges to* x *iff every subsequence* y_m *of* x_n *has subsequence* z_p *such that* $z_p \Rightarrow x$.

Proof. See [Du]. □

It is not hard to show

Lemma 3. *Topology $\bar{\tau}$ equals the quotient topology of τ and the natural mapping is open.*

Corollary 2. *If $\lambda < \underline{b}$ then $(\mathscr{P}(\lambda)/I, \bar{\tau})$ is a Fréchet space.*

By a Maharam submeasure on complete Boolean algebra $\mathscr{A}$, we mean a function $\mu: \mathscr{A} \to [0, 1]$ such that

(a) $\mu(x) = 0$ iff $x = 0$;
(b) if $x \leq y$, then $\mu(x) \leq \mu(y)$;
(c) $\mu(x \vee y) \leq \mu(x) + \mu(y)$; and
(d) if $\bigwedge_{k\in\omega} \bigvee_{n\geq k}(x_n \triangle x) = 0$, then $\lim_{n\to\infty} \mu(x_n) = \mu(x)$.

Similarly, as in the case of measure, we have:

Proposition 3. *Let D be a subspace of the reals of uncountable cardinality $< 2^{\omega}$. Let I be a σ-ideal of $\mathscr{P}(D)$ such that $\mathscr{P}(D)/I$ is ccc. If* MA *holds, there is no Maharam submeasure on $\mathscr{P}(D)/I$.*

Proof. It is sufficient to prove the nonexistence of a nontrivial, nonnegative function μ on $\mathscr{P}(D)$ which is zero on points and satisfies as (b), (c), and (d) as in the definition of Maharam submeasure.

The set D is a Q-set; i.e., every subset of D is G_δ in D. If $Y \subseteq D$, then $Y = \bigcap_{n\in\omega} G_n$, where G_n is a monotonically decreasing sequence of open subsets of D. Then $\mu(G_n) \to \mu(Y)$, and hence $\mu(Y) = 0$ iff, for each $\varepsilon > 0$, there is an open set $G \supseteq Y$ and $\mu(G) < \varepsilon$. Now we may repeat the classical proof of the statement 'MA implies the nonexistence of measure on sets with cardinality $< 2^{\omega}$' (for example see [J, pp. 563–564]). □

Now we can prove the main theorem of this paper.

Theorem 1. *If* Con($ZFC+$ *there exists a measurable cardinal*) *then* Con($ZFC+$ MA $+\neg$ CH $+$ *there exists a complete, weakly (ω, ∞)-distributive, ccc, atomless Boolean algebra without any Maharam submeasure*).

Proof. Let M be a countable transitive model with measurable cardinal κ, and let I be a nonprincipal κ-complete prime ideal over κ. By forcing it to satisfy ccc, we may obtain model $M[G]$ for MA and $\kappa < 2^{\omega}$ (see [Ku]). Then in $M[G]$ ideal J, defined as

$$x \in J \text{ iff } x \subseteq y \text{ for some } y \in I,$$

is a σ-saturated, κ-complete ideal over κ (see [J, p. 425]).

In $M[G]$ the Boolean algebra $\mathscr{P}(\kappa)/J$ is complete, atomless, satisfies ccc and (by Corollary 2 and Lemma 2) Fréchet diagonal condition (L4) for $\Rightarrow$ convergence. This implies weak (ω, ∞)-distributivity of $\mathscr{P}(\kappa)/J$ (see [V2]). By Proposition 3, on $\mathscr{P}(\kappa)/J$ there is no Maharam submeasure. □

Remark 2.

- In the model considered above, there is no Suslin line, because MA $+$ $\neg$ CH is true.

- In the Boolean $\mathscr{P}(\kappa)/J$, there is no finite-additive, strictly positive measure (see Kelley [1959]).
- The method used in the proof of Proposition 3 above shows additionally that $\mathscr{P}(\kappa)/J$ is countably generated.

The weak (ω, ∞)-distributivity of algebra $\mathscr{P}(\kappa)/J$ had been proved independently and by different method by A. Kamburelis.

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Institute of Mathematics, Gdansk University, Wita Stwosza, 80–952 Gdansk, Poland