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REMARKS ON BLOW UP FOR A NONLINEAR PARABOLIC EQUATION WITH A GRADIENT TERM

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ABSTRACT. We consider a nonlinear parabolic equation previously studied by Chipot and Weissler, and Kawohl and Peletier. We give simple sufficient conditions for the presence and absence of L^∞-blow up.

1. INTRODUCTION

This paper was motivated by recent studies of Chipot and Weissler [CW], and Kawohl and Peletier [KP]. They studied the problem

$$u_t = \Delta u - |\nabla u|^q + \lambda u^p, \qquad x \in D,\ t > 0, \tag{1}$$

$$u(x, t) = 0, \qquad x \in \partial D,\ t > 0, \tag{2}$$

$$u(x, 0) = u_0(x) \geq 0, \qquad x \in D, \tag{3}$$

where $D \subset \mathbb{R}^N$ is a smoothly bounded domain, $\lambda > 0$, and p, $q > 1$. Chipot and Weissler proved that blow up occurs for suitable u_0 under the assumptions

$$1 < q < 2p/(p+1), \qquad p < (N+2)/(N-2) \text{ if } N > 2,$$

or

$$q = 2p/(p+1), \qquad N = 1,\ p \text{ is large enough.}$$

We give a simple sufficient condition on the initial value that implies blow up, provided $N = 1$, $p > 1$, and either $1 < q < 2p/(p+1)$ or $q = 2p/(p+1)$. In the case $q = 2p/(p+1)$, we have to assume that λ is large enough. To consider blow up as depending on the value of λ is the point of view taken in [KP], rather than [CW].

A more precise description of our blow up result reads as follows. We show that $u(t, u_0)$ blows up if $u_0 \geq v$, $u_0 \not\equiv v$, and v is the unique positive equilibrium. This is a consequence of a uniform a priori L^∞-estimate of any global increasing solution (established also for $N > 1$). The assumption of

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monotonicity enables us to use a technique from [F] to derive the a priori bound. In [F], similar problems without gradient terms were studied.

In [KP] it is shown that the gradient damping term prevents blow up if $1 < p \leq q = 2$. We give a simple proof of the fact that blow up (in the L^∞-norm) cannot occur if $1 < p \leq q$. If, in addition, $q \leq 2$, then all solutions are global and bounded.

For $p > q = 2$ one may still have L^∞-blow up phenomena. In [KP] it is proved that given u_0 there is a $\lambda^*(u_0) > 0$ such that $u(t, u_0)$ blows up provided $\lambda > \lambda^*$. We derive a lower bound for λ^* in terms of the sup-norm of u_0.

2. Establishing blow up

We adopt the notion of solution from [CW], where local theory in $W_0^{1,s}(D)$ was constructed (s is sufficiently large). We recall the following facts from [CW]. Assume

$$u_0 \in W^{3,s} \cap W_0^{1,s}, \tag{4}$$

$$u_0 \geq 0 \quad \text{in } D, \tag{5}$$

$$\Delta u_0 - |\nabla u_0|^q + \lambda u_0^p = 0 \quad \text{on } \partial D, \tag{6}$$

$$\Delta u_0 - |\nabla u_0|^q + \lambda u_0^p \geq 0 \quad \text{in } D. \tag{7}$$

Then $u(t, u_0) \geq 0$, $u_t(t, u_0) \geq 0$ for all $t \in [0, t_{\max}(u_0))$, $t_{\max}(u_0)$ is the existence time of the maximal solution $u(t, u_0)$. If $q < 2$ and $t_{\max}(u_0) < \infty$, then $|u(t, u_0)|_\infty \to \infty$ as $t \to t_{\max}(u_0)$ ($| \cdot |_r$ denotes the norm in $L^r(D)$, $1 \leq r \leq \infty$). The energy of the solution $u(t, u_0)$,

$$E(u(t, u_0)) := \frac{1}{2}\left|\nabla u(t, u_0)\right|_2^2 - \frac{\lambda}{p+1}\left|u(t, u_0)\right|_{p+1}^{p+1},$$

is a nonincreasing function of $t \in [0, t_{\max}(u_0))$ if u_0 satisfies (4)–(7). More precisely,

$$\int_0^t |u_t(s)|_2^2 \, ds + E(u(t)) \leq E(u_0). \tag{8}$$

We are now prepared to state our first lemma.

Lemma 1. *Let u_0 satisfy (4)–(7); let $t_{\max}(u_0) = \infty$; and assume*

$$q < 2p/(p+1), \qquad \lambda > 0, \tag{9}$$

or

$$q = 2p/(p+1), \qquad \lambda > \bar{\lambda}_p = 2^p(p+1)(p-1)^{-p-1}. \tag{10}$$

Then there is a positive constant $L = L(E(u_0))$ such that

$$|u(t, u_0)|_2 \leq L \qquad \textit{for } t \geq 0.$$

Proof. We proceed analogously as in the proof of Lemma 1.2 in [F]. Obvious manipulations yield that for any real number ρ we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 = -(p+1-\rho)E(u) + \frac{p-1-\rho}{2}|\nabla u|_2^2 + \frac{\lambda\rho}{p+1}|u|_{p+1}^{p+1} - \int_D |\nabla u|^q u. \tag{11}$$

Our aim is to derive a differential inequality for $y(t) := |u(t)|_2^2$. To do this we first estimate $|\nabla u|^q u$ using the well-known inequality:

$$AB \le (\varepsilon A)^r/r + (B/\varepsilon)^s/s, \tag{12}$$

which holds for $A, B, r, s, \varepsilon > 0$, $1/r + 1/s = 1$.

Take $\rho \in (0, p-1)$. Setting $A = |\nabla u|^q$, $B = u$, $r = 2/q$, and $\varepsilon = ((p-1-\rho)/q)^{q/2}$ we obtain

$$\begin{aligned} |\nabla u|^q u &\le \frac{p-1-\rho}{2}|\nabla u|^2 + C_1(p, q, \rho)u^{2/(2-q)}, \\ C_1 &= \frac{2-q}{2}\left(\frac{p-1-\rho}{q}\right)^{q/(q-2)}. \end{aligned} \tag{13}$$

If $q < 2p/(p+1)$, then we choose $0 < \delta < \lambda\rho/(p+1)$ and again we use (12) with $A = u^{2/(2-q)}$, $B = C_1$, $r = (p+1)(2-q)/2$, and $\varepsilon = (r\delta)^{1/r}$ to get

$$|\nabla u|^q u \le \frac{p-1-\rho}{2}|\nabla u|^2 + \delta u^{p+1} + C_2, \tag{14}$$

with C_2 being some positive constant depending on $p, q, \delta, \lambda, \rho$.

If $q = 2p/(p+1)$, then $2/(2-q) = p+1$ and we shall need the inequality

$$C_1(p, \rho) < \lambda\rho/(p+1).$$

To have the range of λ minimally restricted we take $\rho = (p-1)/(p+1)$. This explains the assumption (10). Using (14) (or (13) if $q = 2p/(p+1)$), the inequality $E(u(t)) \le E(u_0)$, and Hölder's inequality we obtain from (11) that

$$y' \ge C_3 y^{(p+1)/2} - C_4, \tag{15}$$

where C_3, C_4 are some positive constants that depend on λ, p, q, and $|D|$—the Lebesgue measure of D, and C_4 depends also on $E(u_0)$. Since u is a global solution and $y(t) = |u(t)|_2^2$ satisfies (15), we see that

$$|u(t)|_2^2 \le (C_4/C_3)^{2/(p+1)} \quad \text{for } t \ge 0.$$

Lemma 2. *Under the assumptions of Lemma* 1 *it holds that*

$$\sup_{t\ge0} |u(t, u_0)|_{p+1} < \infty.$$

Proof. Although the proof is analogous to the proof of Lemma 1.5 in [F], we shall give it here for the reader's convenience. It is based on the classical concavity method introduced in [L]. Assume that $|u(t, u_0)|_{p+1}$ is unbounded. Then

$$|u(t, u_0)|_{p+1} \to \infty \quad \text{as } t \to \infty \tag{16}$$

since $|u(\cdot, u_0)|_{p+1}$ is nondecreasing.

Put

$$M(t) = \int_0^t |u|_2^2.$$

If $q < 2p/(p+1)$ we use (14) and (8) to derive the following inequality from (11):

$$\frac{1}{2}M''(t) \geq (p+1-\rho)\left(\int_0^t \int_D u_t^2 - E(u_0)\right) + \left(\frac{\lambda\rho}{p+1} - \delta\right)|u(t)|_{p+1}^{p+1} - |D|C_\delta.$$

Writing

$$\begin{aligned} M'^2(t) &= \left(\int_0^t \frac{d}{dt}|u|_2^2 + |u_0|_2^2\right)^2 \\ &= 4\left(\int_0^t \int_D uu_t\right)^2 + 2M'|u_0|_2^2 - |u_0|_2^4, \end{aligned}$$

we obtain

$$\begin{aligned} &MM'' - \frac{p+1-\rho}{2}M'^2 \\ &\quad \geq 2(p+1-\rho)\left(\int_0^t \int_D u^2 \int_0^t \int_D u_t^2 - \left(\int_0^t \int_D uu_t\right)^2\right) \\ &\qquad + 2M\left(\left(\frac{\lambda\rho}{p+1} - \delta\right)|u(t)|_{p+1}^{p+1} - (p+1-\rho)E(u_0) - |D|C_\delta\right) \\ &\qquad - (p+1-\rho)M'|u_0|_2^2 + (p+1-\rho)|u_0|_2^4/2. \end{aligned}$$

The first term on the right-hand side is nonnegative according to the Schwarz inequality; the second one tends to infinity by (16); and the third one is bounded according to Lemma 1. Hence, there is a $t_0 \geq 0$ such that the right-hand side is positive for $t \geq t_0$. Therefore $(M^{-\mu})'' < 0$ for $t \geq t_0$, where $\mu = (p+1-\rho)/2 > 0$. Since $M^{-\mu}$ is decreasing, it must have a root, a contradiction.

If $q = 2p/(p+1)$ we proceed similarly. The only difference is that we use (13) instead of (14).

Theorem 1. *Let* (4)–(7) *and* (9) *or* (10) *hold. Assume further that* $t_{\max}(u_0) = \infty$ *and*

$$(17) \qquad p < (N+2)/(N-2) \quad \text{if } N > 2.$$

Then there is a constant $K = K(|u_0|_\infty, E(u_0))$ *such that* $|u(t, u_0)|_\infty \leq K$ *for* $t \geq 0$. *Moreover,* $u(t, u_0) \to v$ *in* $W_0^{1,s}(D)$ *as* $t \to \infty$; v *is a stationary solution.*

Proof. If u is a nonnegative function that satisfies

$$u_t \leq \Delta u + \lambda u^p \quad \text{in } D\times(0,\infty), \qquad u = 0 \quad \text{on } \partial D\times(0,\infty), \quad u(\cdot, 0) \in L^\infty(D),$$

then it is known (see e.g. [R, Proposition 2]) that boundedness of u in L^r implies boundedness in L^∞ provided $r > (p-1)\max\{1, N/2\}$. More precisely,

$$|u(t, u_0)|_\infty \leq C(|u_0|_\infty, \sup_{0\leq s\leq t} |u(s, u_0)|_r) \quad \text{for } t \geq 0. \tag{18}$$

Lemma 2 and (18) with $r = p+1$ yield that

$$\sup_{t\geq 0} |u(t, u_0)|_\infty < \infty.$$

For $q \leq 2$, we have the estimate

$$\|u\|_{C^{1+\mu, (1+\mu)/2}(\overline{D}\times[0,T])} \leq \gamma(\|u\|_{C(\overline{D}\times[0,T])}, \|u_0\|_{C^2(\overline{D})}) \tag{19}$$

for any $\mu \in (0, 1)$ and $T > 0$ (see [A, Theorem 2.2]). Therefore, $\{u(t, u_0) : t \geq 0\}$ is precompact in $W_0^{1,s}(D)$. Due to monotonicity $u(t, u_0) \to v$, v is a stationary solution.

Similarly as in the proof of Lemma 1.8 in [F], we derive an a priori bound of $|v|_{p+1}$ in terms of $E(u_0)$. For v it holds that

$$|\nabla v|_2^2 = \lambda |v|_{p+1}^{p+1} - \int_D |\nabla v|^q v;$$

therefore,

$$\begin{aligned}
&(p+1-\rho)E(u_0) \\
&\quad > (p+1-\rho)E(v) = \frac{p-1-\rho}{2}|\nabla v|_2^2 + \frac{\lambda\rho}{p+1}|v|_{p+1}^{p+1} - \int_D |\nabla v|^q v \\
&\quad \geq (\lambda\rho/(p+1) - \delta)|v|_{p+1}^{p+1} - |D|C_\delta \quad \text{if } q < 2p/(p+1); \\
&\quad (\geq (\lambda\rho/(p+1) - C_1)|v|_{p+1}^{p+1} \quad \text{if } q = 2p/(p+1).)
\end{aligned}$$

The last inequality follows from (14) (or (13)).

The monotonicity of $|u(\cdot, u_0)|_{p+1}$ together with (18) now implies the desired result.

Before we state the next theorem, let us recall some known results (proved in [CW]) on the one-dimensional stationary problem. If (9) holds or if $q = 2p/(p+1)$, $\lambda > \lambda_p = (2p)^p(p+1)^{-2p-1}$ (notice that $\overline{\lambda}_p > \lambda_p$), then there is a unique positive stationary solution v (for any interval D). From Lemma 4.7 and Proposition 5.9 in [CW], it follows that it is unstable, because it is possible to construct supersolutions arbitrarily close to v from below and subsolutions arbitrarily close to v from above.

Theorem 2. *Let* (9) *or* (10) *hold. Assume that* $N = 1$, $u_0 \in W_0^{1,s}(D)$, $u_0 \geq v$, *and* $u_0 \not\equiv v$, *where* v *is the unique positive stationary solution. Then* $u(t, u_0)$ *blows up in a finite time* (*in the* L^∞*-norm*).

Proof. Since v is unstable, there is a solution w of (1), (2) on $\overline{D} \times (-\infty, 0]$ such that $w_t > 0$, $w > v$ in $D \times (-\infty, 0]$ and $w \to v$ in $C^2(\overline{D})$ as $t \to -\infty$ (see [M, Theorem 1]). Hence $t_{\max}(w(\cdot, 0)) < \infty$; otherwise $u(t, w(\cdot, 0))$

would tend to a stationary solution $z > v$ (by Theorem 1). The maximum principle implies the existence of $t_1 < 0 \le t_2$ for which $w(\cdot, t_1) \le u(t_2, u_0)$ and the conclusion follows.

3. Excluding blow up

Theorem 3. *Assume that* $u_0 \in W_0^{1,s}(D)$, $u_0 \ge 0$.

(i) *If* $q \ge p > 1$, *then there is a constant* $K_1 = K_1(|u_0|_\infty) > 0$ *such that* $u(t, u_0) \le K_1$ *for* $t \in [0, t_{\max}(u_0))$.

(ii) *If* $p > q \ge 2p/(p+1)$, *then there are positive numbers* $\tilde{\lambda} = \tilde{\lambda}(|u_0|_\infty)$, $K_2 = K_2(|u_0|_\infty)$ *such that* $u(t, u_0) \le K_2$ *for* $t \in [0, t_{\max}(u_0))$, *provided* $0 < \lambda < \tilde{\lambda}$.

Proof. Suppose first for simplicity that $N = 1$, $D = (1, a)$, and $a > 1$. Then

$$U(x) := \alpha^{2/(p-1)} e^{\alpha x}$$

is a suitable supersolution.

In the case (i) it suffices to choose

$$\alpha = \max\{|u_0|_\infty^{(p-1)/2}, (1+\lambda)^\beta\}, \qquad \beta = \frac{p-1}{q(p+1)-2p}.$$

In the case (ii) take

$$\alpha = \max\{1, |u_0|_\infty^{(p-1)/2}\}, \qquad \tilde{\lambda} = e^{a\alpha(q-p)}(\alpha^{1/\beta} - e^{\alpha(1-q)}).$$

In both cases we have

$$\begin{aligned} &U''(x) - (U'(x))^q + \lambda U^p(x) \\ &\quad = e^{\alpha p x b} \alpha^{2p/(p-1)}(e^{\alpha(1-p)x} - \alpha^{1/\beta} e^{\alpha(q-p)x} + \lambda) \le 0; \end{aligned}$$

hence by the maximum principle $u(t, u_0) \le U$ for $0 \le t < t_{\max}(u_0)$.

For a general domain D in $\mathbb{R}^N$, we set

$$U(x) = \alpha^{2/(p-1)} \exp\left(\alpha \sum_{i=1}^N x_i\right)$$

and we may assume without loss of generality that $\sum_{i=1}^N x_i \in (1, a)$ for $x = (x_1, \dots, x_N) \in D$.

Remark. If $q \le 2$, then $t_{\max}(u_0) = \infty$ in Theorem 3, due to (19). This implies that $u(t, u_0) \to 0$ as $t \to \infty$ for any $u_0 \in W_0^{1,s}$, $u_0 \ge 0$, provided $1 < p \le q \le 2$ and D is a ball with a sufficiently small radius. Under these circumstances, no positive stationary solutions exist (see [CW, Corollary 5.4 (i)]).

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