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CUSP FORMS ASSOCIATED TO RANK 2 PARABOLIC SUBGROUPS OF KLEINIAN GROUPS

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ABSTRACT. To each rank 2 parabolic subgroup of a Kleinian group Γ, we associate a cusp form and a linear functional on the Eichler cohomology space $PH^1(\Gamma, \Pi_{2q-2})$. We explore the relation between these two objects and as a consequence we evaluate certain Poincaré series for rank 2 parabolic groups.

This paper is a continuation of the study of automorphic forms associated to elements of Kleinian groups. We treated the case of cusp forms associated to loxodromic elements in [K3] and the holomorphic forms associated to parabolic elements in [K1]. We combine ideas from [K3] and [K1] to investigate the cusp forms associated to rank 2 parabolic subgroups of Kleinian groups.

Let A and B be generators of a rank 2 parabolic subgroup G of a Kleinian group Γ. We associate (in §2) to the pair (A, B) a linear functional

$$l_{A,B} : PH^1(\Gamma, \Pi_{2q-2}) \to \mathbb{C}$$

on the space of parabolic cohomology classes for Γ,[1] and (in §3) a cusp form $\varphi_{A,B} \in \mathbb{A}_q(\Omega, \Gamma)$. One of the main results is (Theorem 4.3) the connection between these two invariants provided by the Bers embedding

$$\beta^* : \mathbb{A}_q(\Omega, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2})$$

and the Petersson scalar product

$$\langle \cdot, \cdot \rangle : \mathbb{A}_q(\Omega, \Gamma) \times \mathbb{A}_q(\Omega, \Gamma) \to \mathbb{C}.$$

A by-product of this work is the evaluation of some naturally occurring elliptic sums. See §5.

1. INVARIANTS FOR PARABOLIC ELEMENTS

1.1. A parabolic Möbius transformation $A \in PSL(2, \mathbb{C})$ has a unique fixed point $\alpha = \alpha(A) \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and a well-defined translation length

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[1]We will use the notation and terminology of [K2], [K1] and [K3].

$\tau = \tau(A) \in \mathbb{C}^* = \mathbb{C} - \{0\}$. In normal form:

$$A(z) = z + \tau, \qquad A = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \quad \text{if } \alpha = \infty,$$

and

$$\frac{1}{A(z) - \alpha} = \frac{1}{z - \alpha} + \tau, \qquad A = \begin{bmatrix} 1 + \alpha\tau & -\alpha^2\tau \\ \tau & 1 - \alpha\tau \end{bmatrix} \quad \text{if } \alpha \in \mathbb{C}.$$

For future use, we observe that for finite α,

$$A'(\alpha) = 1 \quad \text{and} \quad A''(\alpha) = -2\tau.$$

If C is an arbitrary Möbius transformation, then

$$\alpha(C \circ A \circ C^{-1}) = C(\alpha(A)) \quad \text{and} \quad \tau(C \circ A \circ C^{-1}) = \tau(A)/C'(\alpha). \tag{1.1.1}$$

The above formulae must be properly interpreted for α or $C(\alpha) = \infty$. Assume, for example, that $\alpha \in \mathbb{C}$ and $C(\alpha) = \infty$. Write

$$C = \begin{bmatrix} \beta & \delta \\ 1 & -\alpha \end{bmatrix} \in PGL(2, \mathbb{C}).$$

Then $C'(z) = -(\alpha\beta + \delta)/((z - \alpha)^2)$ and we must interpret $C'(\alpha)$ as $1/(\alpha\beta + \delta)$.

1.2. The holomorphic differential form on $\hat{\mathbb{C}} - \alpha$ for the cyclic group $\langle A \rangle$ is defined by

$$h(z) = h_A(z) = \frac{1}{\tau(A)} \quad \text{if } \alpha = \infty,$$

and

$$h(z) = h_A(z) = \frac{1}{\tau(A)} \frac{-1}{(z - \alpha)^2} \quad \text{if } \alpha \in \mathbb{C}.$$

The function h is canonically associated to the Möbius transformation A and is characterized by the properties

$$\int_{z_0}^{A(z_0)} h(z)dz = 1, \quad \text{all } z_0 \in \hat{\mathbb{C}} - \{\alpha\}$$

(as long as the path of integration stays in $\hat{\mathbb{C}} - \{\alpha\}$), and $h(z)dz$ has simple poles at the two punctures of $(\hat{\mathbb{C}} - \{\alpha\})/\langle A \rangle$.

As a consequence of (1.1.1) and the basic identity[2]

$$C(z) - C(\zeta) = (z - \zeta)C'(z)^{1/2}C'(\zeta)^{1/2}, \tag{1.2.1}$$

which is valid for all $C \in PSL(2, \mathbb{C})$ and all z and $\zeta \in \hat{\mathbb{C}}$, we have

$$h_{C \circ A \circ C^{-1}}(C(z))C'(z) = h_A(z), \quad \text{all } z \in \hat{\mathbb{C}}, \tag{1.2.2}$$

or equivalently,

$$C_q^*(h^q_{C \circ A \circ C^{-1}}) = h_A^q, \quad \text{all } q \in \mathbb{Z}. \tag{1.2.3}$$

[2]This formula (and similar formulae) must be adjusted if z, ζ, $C(z)$, or $C(\zeta)$ equals ∞. We leave such modifications to the reader (generally).

1.3. The polynomial ring of A acts on the right on the vector space Π_{2q-2} of polynomials of degree $\leq 2q-2$ by the Eichler action (from now on $q \geq 2$). The kernel of the operator $[A - I]$ is the one-dimensional space with basis h_A^{1-q}. The image of $[A - I]$ consists of

$$\Pi_{2q-2}(\alpha) = \{v \in \Pi_{2q-2}\,;\, v(\alpha) = 0\}.$$

For details see [K2, §8.3].[3]

2. Linear functionals on $PH^1(\Gamma, \Pi_{2q-2})$ associated to rank 2 parabolic subgroups of Γ

2.1. Let (A, B) be an ordered pair of generators for a rank 2 parabolic subgroup G of a Kleinian group Γ. Let χ_1 be a cocycle representing a cohomology class in $PH^1(\Gamma, \Pi_{2q-2})$. Choose $v \in \Pi_{2q-2}$ such that

$$\chi_1(A) = v \cdot A - v.$$

Then the cocycle

$$\chi(\gamma) = \chi_1(\gamma) - v \cdot \gamma + v, \qquad \gamma \in \Gamma,$$

is cohomologous to χ_1 and satisfies

$$\chi(A) = 0.$$

The cocycle relation

$$\chi(A) \cdot B + \chi(B) = \chi(A \circ B) = \chi(B \circ A) = \chi(B) \cdot A + \chi(A)$$

is a consequence of the fact that A and B commute. It implies that $\chi(B) \in \operatorname{kernel}[A - I]$. Thus there exists a $l(\chi) \in \mathbb{C}$ such that

$$\chi(B) = l(\chi) h_A^{1-q}.$$

Proposition. (a) *The rule*

$$\chi \longmapsto l(\chi) = l_{A,B}(\chi)$$

defines an element of the dual space $PH^1(\Gamma, \Pi_{2q-2})^*$ *of* $PH^1(\Gamma, \Pi_{2q-2})$.

(b) *For all* $C \in \Gamma$,

$$l_{C \circ A \circ C^{-1}, C \circ B \circ C^{-1}} = l_{A,B}.$$

Proof. Let χ_1 be an arbitrary cocycle with $\chi_1(A) = 0$ and χ_1 cohomologous to χ. Then there exists a $v \in \Pi_{2q-2}$ such that

$$\chi_1(\gamma) = \chi(\gamma) - v \cdot \gamma + v, \qquad \gamma \in \Gamma.$$

It follows that $v \in \operatorname{kernel}[A - I] = \operatorname{kernel}[B - I]$. Thus $\chi_1(B) = \chi(B)$ and l is well defined.

2.2. For arbitrary $C \in PSL(2, \mathbb{C})$, the linear map

$$\wedge : PH^1(\Gamma, \Pi_{2q-2}) \to PH^1(C\Gamma C^{-1}, \Pi_{2q-2})$$

[3]There are a number of misprints in [K2]. For example, the formula for Image$(E - 1)$ in §8.3 should read Image $(E - 1) = \{p \in \Pi_{2q-2}\,;\, p(a) = 0\}$.

defined on the cocycle level by

$$\hat{\chi}(C \circ \gamma \circ C^{-1}) = \chi(\gamma) \cdot C^{-1}, \qquad \gamma \in \Gamma,$$

is a (surjective) isomorphism. For $\chi \in PZ^1(\Gamma, \Pi_{2q-2})$ with

$$\chi(A) = 0 \quad \text{and} \quad \chi(B) = l(\chi) h_A^{1-q},$$

we have

$$\hat{\chi}(C \circ A \circ C^{-1}) = 0 \quad \text{and} \quad \hat{\chi}(C \circ B \circ C^{-1}) = l(\chi) h_{C \circ A \circ C^{-1}}^{1-q};$$

which shows that

$$l_{A,B}(\chi) = l_{C \circ A \circ C^{-1}, C \circ B \circ C^{-1}}(\hat{\chi}). \tag{2.2.1}$$

2.3. If $C \in \Gamma$, then for every cocycle χ for Γ,

$$\chi(C \circ \gamma \circ C^{-1}) = \chi(\gamma) \cdot C^{-1} + \chi(C) \cdot C^{-1} \cdot [C \circ \gamma \circ C^{-1} - I]$$

for all $\gamma \in \Gamma$. Thus χ is cohomologous to $\hat{\chi}$ and we conclude that

$$l_{C \circ A \circ C^{-1}, C \circ B \circ C^{-1}}(\chi) = l_{C \circ A \circ C^{-1}, C \circ B \circ C^{-1}}(\hat{\chi}) = l_{A,B}(\chi).$$

We have completed the proof of Proposition 2.1.

2.4. Let (A_1, B_1) be a second ordered pair of generators for the group G. Then, as is well known, there exist integers a, b, c, d so that

$$A_1 = A^a \circ B^b, \quad B_1 = A^c \circ B^d \quad \text{and} \quad ad - bc = \varepsilon = \pm 1.$$

We are interested in comparing l_{A_1, B_1} with $l_{A,B}$. As a consequence of (2.2.1), we may assume that $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & \tau \\ 0 & 1 \end{smallmatrix}\right]$ with $\operatorname{Im} \tau \neq 0$. Thus

$$A_1 = \begin{bmatrix} 1 & a + b\tau \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & c + d\tau \\ 0 & 1 \end{bmatrix}.$$

Let us choose $\chi \in PZ^1(\Gamma, \Pi_{2q-2})$ with

$$\chi(A) = 0 \quad \text{and} \quad \chi(B) = l(\chi).$$

Then

$$\chi(A_1) = \chi(B^b) = \chi(B) \cdot [B^{b-1} + \cdots + I] = bl(\chi), \quad \chi(B_1) = dl(\chi).$$

We must replace the cocycle χ by a cohomologous cocycle that vanishes at A_1. Let

$$v(z) = \frac{bl(\chi)}{a + b\tau} z, \qquad z \in \mathbb{C}.$$

The cocycle

$$\chi_1(\gamma) = \chi(\gamma) - v \cdot \gamma + v, \qquad \gamma \in \Gamma,$$

vanishes at A_1 and has value

$$\chi_1(B_1) = \frac{\varepsilon}{a + b\tau} l(\chi).$$

It follows that for all $\chi \in PH^1(\Gamma, \Pi_{2q-2})$,

$$l_{A_1, B_1}(\chi) = \frac{\varepsilon}{(a+b\tau)^q} l_{A,B}(\chi).$$

Corollary. *The projective class of the linear functional* $l_{A,B} \in PH^1(\Gamma, \Pi_{2q-2})^*$ *depends only on the conjugacy class of* G *in* Γ.

2.5. Let Δ denote the unit disk, and

$$\Phi : \Delta \to \mathrm{Hom}^{\mathrm{par}}(\Gamma, PSL(2, \mathbb{C}))$$

be a normalized holomorphic family of parabolic homomorphisms on Δ; this means that for each $t \in \Delta$, $\Phi(t)$ is a homomorphism of the Kleinian group Γ into $PSL(2, \mathbb{C})$,

$$\gamma(t) = \Phi(t)(\gamma)$$

defines a holomorphic map from Δ to $PSL(2, \mathbb{C})$ for all $\gamma \in \Gamma$ and is parabolic or the identity whenever $\gamma \in \Gamma$ is parabolic, and $\Phi(0)$ is the identity isomorphism. This normalized family Φ defines a cocycle χ for $q = 2$ by

$$\chi(\gamma)(z) = \dot{\gamma}(z)/\gamma'(z), \qquad z \in \mathbb{C},$$

where

$$\dot{\gamma}(z) = \frac{\partial}{\partial t}\gamma(t)(z)|_{t=0}, \qquad \gamma \in \Gamma, \ z \in \mathbb{C}.$$

For details, see [GK]. If there exists a holomorphic function C from a nonempty neighborhood $\{t \in \mathbb{C}; |t| < \varepsilon\}$ of 0 in $\mathbb{C}$ into $PSL(2, \mathbb{C})$ with $C(0) = I$ and

$$\gamma(t) = C(t) \circ \gamma \circ C(t)^{-1}, \quad \text{all } \gamma \in \Gamma, \ \text{all } |t| < \varepsilon,$$

then χ is a coboundary.

If A and B generate a rank 2 parabolic subgroup of Γ, then for sufficiently small $|t|$, we can define a holomorphic function by

$$\tau(t) = \frac{\tau(B(t))}{\tau(A(t))}.$$

Proposition. *We have*

$$\tau'(0) = l_{A,B}(\chi).$$

Proof. By conjugation, we may assume that $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & \tau_0 \\ 0 & 1 \end{smallmatrix}\right]$ with $\operatorname{Im} \tau_0 \neq 0$. Write

$$A(t) = \begin{bmatrix} 1 + a(t) & b(t) \\ -\frac{a^2(t)}{b(t)} & 1 - a(t) \end{bmatrix}$$

with a and b holomorphic functions with $a(0) = 0$, $b(0) = 1$. It follows that

$$\alpha(A(t)) = -\frac{b(t)}{a(t)}.$$

Since $\alpha(B(t)) = \alpha(A(t))$, it also follows that

$$B(t) = \begin{bmatrix} 1 + \beta(t)\frac{a(t)}{b(t)} & \beta(t) \\ -\frac{a^2(t)\beta(t)}{b^2(t)} & 1 - \beta(t)\frac{a(t)}{b(t)} \end{bmatrix}$$

with β some holomorphic function satisfying $\beta(0) = \tau_0$. Let

$$C(t) = \begin{bmatrix} b(t)^{-1/2} & 0 \\ a(t)b(t)^{-1/2} & b(t)^{1/2} \end{bmatrix}.$$

Then

$$C(t) \circ A(t) \circ C(t)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$C(t) \circ B(t) \circ C(t)^{-1} = \begin{bmatrix} 1 & \beta(t)/b(t) \\ 0 & 1 \end{bmatrix}.$$

Since conjugation preserves the trace and does not change the cohomology class, it suffices to assume that

$$A(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 1 & \tau(t) \\ 0 & 1 \end{bmatrix}$$

with τ a holomorphic function in some neighborhood of 0 with $\tau(0) = \tau_0$. It follows that $\chi(A) = 0$ and $\chi(B) = \tau'(0) = l_{A,B}(\chi)$.

3. Cusp forms associated to rank 2 parabolic subgroups of Kleinian groups

3.1. Let A and B generate a rank 2 parabolic subgroup G of the Kleinian group Γ. Then

$$\varphi_{A,B}(z) = \sum_{\gamma \in G \backslash \Gamma} h_A(\gamma z)^q \gamma'(z)^q, \qquad z \in \Omega,$$

defines a cusp form for Γ (that is, an element of $\mathbb{A}_q(\Omega, \Gamma)$; here Ω is the region of discontinuity of Γ).

3.2. For arbitrary $C \in \Gamma$, we have

$$\varphi_{C \circ A \circ C^{-1}, C \circ B \circ C^{-1}} = \varphi_{A,B}$$

as a consequence of (1.2.3).

3.3. The cusp form $\varphi_{A,B}$ is related to a holomorphic Eichler integral. Assume that Γ is a finitely generated Kleinian group. Then

$$\psi_{A,B}(z) = \sum_{\gamma \in \Gamma/G} \frac{\gamma'(\alpha)^q}{\gamma(\alpha) - z}, \qquad z \in \Omega,$$

(here $\alpha = \alpha(A)$) defines a holomorphic Eichler integral (that is, an element of $\mathbb{E}^c_{1-q}(\Omega, \Gamma)$), with

$$\begin{aligned} \frac{d^{2q-1}}{dz^{2q-1}} \psi_{A,B}(z) &= (2q-1)! \sum_{\gamma \in \Gamma/G} \frac{\gamma'(\alpha)^q}{(\gamma(\alpha) - z)^{2q}} \\ &= (2q-1)! \sum_{\gamma \in G \backslash \Gamma} \frac{\gamma'(z)^q}{(\gamma(z) - \alpha)^{2q}} \\ &= (-1)^q \tau(A)^q (2q-1)! \varphi_{A,B}. \end{aligned}$$

The Eichler integral ψ is *not* parabolic with respect to A. It satisfies

$$(\mathrm{pd}\,\psi)(A)(\alpha) = (q - \tfrac{1}{2})A''(\alpha) = (1 - 2q)\tau(A),$$
$$(\mathrm{pd}\,\psi)(B)(\alpha) = (1 - 2q)\tau(B).$$

The above formulae assume that $\langle A, B\rangle$ is the stabilizer of α in Γ. Adjustments need to be made when $\langle A, B\rangle$ is a proper subgroup of the stabilizer. The cohomology class $\mathrm{pd}\,\psi$ is parabolic with respect to every parabolic element that fixes a point *not* Γ-equivalent to α. For details, see [K1].

3.4. As a consequence of §3.3, we see that $\varphi_{A,B} \neq 0$ provided α is accessible (defined in [K1, §2]) and $\langle A, B\rangle$ is the stabilizer of α in Γ. As a matter of fact, under this hypothesis, whenever A (or B) fixes a component Δ of Ω, then $\varphi_{A,B}|\Delta \neq 0$. More generally, let α be an accessible limit point of Γ whose stabilizer $\Gamma(\alpha)$ contains a maximal rank 2 parabolic subgroup $P(\alpha)$ generated by A and B as above. Then (see [K1, §0]) $\Gamma(\alpha)/P(\alpha)$ is a cyclic group of order $n = 1, 2, 3, 4$ or 6. The fixed point α is *q-admissible* if $q \equiv 0 \pmod{n}$. We showed in [K1, §5] that $\varphi_{A,B} \neq 0$ if and only if α is q-admissible.

4. Linear functionals on cusp forms associated to rank 2 parabolic subgroups

4.1. Let Γ be a nonelementary Kleinian group. The Bers map

$$\beta^* : \mathbb{A}_q(\Omega, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2})$$

is discussed in [K2, §0.2]. For $\psi \in \mathbb{A}_q(\Omega, \Gamma)$, we let F be a potential for $\mu = \lambda^{2-2q}\overline{\psi}$ (here λ is the Poincaré metric on the region of discontinuity Ω of Γ) and $\beta^*(\psi) = \mathrm{pd}\,F$. We need to construct specific potentials for μ.

4.2. Let A be a parabolic element of Γ. Let F be a potential for μ that vanishes at $A^j(a)$, $j = 0, 1, \ldots, 2q-2$, where $a \in \hat{\mathbb{C}}$ is arbitrary with $a \neq \alpha = \alpha(A)$. We claim that $(\mathrm{pd}\,F)(A) = 0$. The polynomial $(\mathrm{pd}\,F)(A) = F \cdot A - F$ automatically vanishes at α and at $A^j(a)$, $j = 0, \ldots, 2q-3$. It is thus identically zero. Thus we see that

$$F(A(z))A'(z)^{1-q} - F(z) = 0, \quad \text{all } z \in \hat{\mathbb{C}}. \tag{4.2.1}$$

It follows that $F(\alpha) = 0$ since by induction $F(A^j(a)) = 0$ for all $j \in \mathbb{Z}$ and $\lim_{j\to\infty} A^j(a) = \alpha$ (the potential F is a continuous function).

As a consequence, the potential F_1 for μ that vanishes at α and at $A^j(a)$, $j = 0, \ldots, 2q-3$, agrees with F. We have constructed two potentials

$$F(z) = \frac{p(z)}{2\pi i}\iint_\Omega \frac{\mu(\zeta)}{(\zeta - z)p(\zeta)}\, d\zeta \wedge d\overline{\zeta}, \qquad z \in \hat{\mathbb{C}} \quad ,$$

that satisfy (4.2.1) by choosing

$$p(z) = \prod_{j=0}^{2q-2}(z - A^j(a)) \quad \text{or} \quad p(z) = (z - \alpha)\prod_{j=0}^{2q-3}(z - A^j(a)).$$

4.3. We can now formulate the following

Theorem. *Let A and B generate a rank 2 parabolic subgroup of the Kleinian group Γ. Then for all $\psi \in \mathbb{A}_q(\Omega, \Gamma)$,*

$$l_{A,B}(\beta^*(\psi)) = -i\langle \varphi_{A,B}, \psi \rangle.$$

Proof. Let $a \in \hat{\mathbb{C}}$, $a \neq \alpha = \alpha(A)$. Let F be the potential for $\mu = \lambda^{2-2q}\overline{\psi}$ that vanishes at $A^j(a)$, $j = 0, \dots, 2q-2$. Then, as observed above, for $\chi = \mathrm{pd}\, F$, $l = l_{A,B}$, we have $\chi(A) = 0$, and hence

$$\chi(B)(z) = l(\chi)(-\tau(A))^{q-1}(z-\alpha)^{2q-2}, \qquad z \in \hat{\mathbb{C}}.$$

Let ω be a standard fundamental domain for $\langle A, B\rangle$ such that $\partial\omega$ consists of oriented sides c_1, c_2, $-A(c_1)$, $-B(c_2)$ and the winding number of $\partial\omega$ around α is -1. We compute

$$\left(\frac{-1}{\tau(A)}\right)^q \int_{\partial\omega} \frac{F(z)dz}{(z-\alpha)^{2q}} = \left(\frac{-1}{\tau(A)}\right)^q \left[\int_{c_1} - \int_{A(c_1)} + \int_{c_2} - \int_{B(c_2)}\right] \frac{F(z)dz}{(z-\alpha)^{2q}}.$$

Now

$$\int_{c_1} \frac{F(z)dz}{(z-\alpha)^{2q}} = \int_{c_1} \frac{F(Az)A'(z)^{1-q}dz}{(Az-\alpha)^{2q}A'(z)^{-q}} = \int_{A(c_1)} \frac{F(z)dz}{(z-\alpha)^{2q}},$$

and

$$\begin{aligned}\int_{c_2} \frac{F(z)dz}{(z-\alpha)^{2q}} &= \int_{c_2} \frac{F(Bz)B'(z)^{1-q} - \chi(B)(z)}{(z-\alpha)^{2q}} dz \\ &= \int_{B(c_2)} \frac{F(z)dz}{(z-\alpha)^{2q}} - l(\chi)(-\tau(A))^{q-1} \int_{c_2} \frac{dz}{(z-\alpha)^2}.\end{aligned}$$

We conclude that

$$\left(\frac{-1}{\tau(A)}\right)^q \int_{\partial\omega} \frac{F(z)dz}{(z-\alpha)^{2q}} = -l(\chi).$$

On the other hand ($G = \langle A, B\rangle$)

$$\begin{aligned}\left(\frac{-1}{\tau(A)}\right)^q \int_{\partial\omega} \frac{F(z)dz}{(z-\alpha)^{2q}} &= -\left(\frac{-1}{\tau(A)}\right)^q \iint_{\omega} \frac{\partial}{\partial\overline{z}}\left[\frac{F(z)}{(z-\alpha)^{2q}}\right] dz \wedge d\overline{z} \\ &= -\left(\frac{-1}{\tau(A)}\right)^q \iint_{\Omega/G} \frac{\lambda^{2-2q}(z)\overline{\psi(z)}}{(z-\alpha)^{2q}} dz \wedge d\overline{z} \qquad (4.3.1) \\ &= -\iint_{\Omega/\Gamma} \varphi_{A,B}(z)\lambda^{2-2q}(z)\overline{\psi(z)} dz \wedge d\overline{z} \\ &= i\langle \varphi_{A,B}, \psi\rangle.\end{aligned}$$

For a justification of the use of Stokes' theorem, see [K3, §2.4].

4.4. Let us assume that Γ is finitely generated and use the notation of §3.4. We have obtained the

Corollary. *The natural restrictions*

$$PH^1(\Gamma, \Pi_{2q-2}) \to PH^1(\Gamma(\alpha), \Pi_{2q-2})$$

and

$$H^1(\Gamma, \Pi_{2q-2}) \to H^1(\Gamma(\alpha), \Pi_{2q-2})$$

are surjective provided α *is accessible.*

We note that we can replace $\Gamma(\alpha)$ by $\mathrm{P}(\alpha)$ if and only if α is q-admissible. For α not q-admissible, every cohomology class for Γ restricts to a coboundary for $\Gamma(\alpha)$ (and thus also for $\mathrm{P}(\alpha)$). Example D of [K1, §7] shows that the assumption that α be accessible cannot be dropped.

5. Some formulae for rank 2 parabolic subgroups

5.1. Let G be a subgroup of Γ. We denote by $\Theta_{G\backslash\Gamma}$ the relative Poincaré series operator mapping q-forms for G to q-forms for Γ. We abbreviate $\Theta_{\{I\}\backslash\Gamma}$ by Θ_Γ and observe that

$$\Theta_\Gamma = \Theta_{G\backslash\Gamma^0}\Theta_G.$$

5.2. We consider the special case of the rank 2 parabolic group G generated by

$$A(z) = z + 1, \quad B(z) = z + \tau, \qquad z \in \hat{\mathbb{C}}, \ \operatorname{Im}\tau \neq 0.$$

For $k \in \mathbb{Z}^+$, we define

$$g_\tau(z) = \frac{1}{2\pi i}\frac{\tau(\tau-1)\cdots(\tau-k)}{(z-\tau)z(z-1)\cdots(z-k)}, \qquad z \in \mathbb{C},$$

and

$$\Theta g_\tau(z) = \sum_{\gamma\in G} g_\tau(\gamma(z)), \qquad z \in \mathbb{C}.$$

We observe that Θg_τ is a Poincaré series for the group G with $q = \frac{1}{2}(k+2) \geq 2$ for k even and $q = \frac{1}{2}(k+3) \geq 2$ for k odd. Thus we conclude quite easily that

$$\Theta g_\tau = (\Theta_G)(g_\tau)$$

converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and defines a doubly periodic function

$$\Theta g_\tau(z+1) = \Theta g_\tau(z) = \Theta g_\tau(z+\tau), \quad \text{all } z \in \mathbb{C},$$

for G. It is most convenient to regard Θg_τ as an automorphic form (for appropriate q, as above). Since Θg_τ has at most simple poles at the lattice points, it must be a constant $c(\tau)$ that depends, a priori, on τ and k.

Theorem. *For all* $\tau \in \mathbb{C}$ *with* $\operatorname{Im}\tau \neq 0$, $c(\tau) = 1$.

Proof. There exist groups Γ (see [K1, §7]) with $G \subset \Gamma$ and $\dim \mathbb{A}_q(\Omega, \Gamma) > 0$. Let us assume that k is even, $k = 2q - 2$ for some integer $q \geq 2$. We will obtain an alternate formula for $l_{A,B}(\beta^* \psi)$ with $\psi \in \mathbb{A}_q(\Omega, \Gamma)$. Let F be the potential for $\lambda^{2-2q}\overline{\psi}$ that vanishes at $A^j(0) = j$, $j = 0, 1, \ldots, 2q-2$. Then (as usual, $l = l_{A,B}$, $\chi =$ pd F)

$$F(A(z)) - F(z) = 0, \quad \text{all } z \in \mathbb{C} \quad ,$$

and

$$F(B(z)) - F(z) = l(\chi), \quad \text{all } z \in \mathbb{C}.$$

It follows that

$$F(\tau) = l(\chi).$$

But

$$\begin{aligned} F(\tau) &= \frac{\tau(\tau-1)\cdots(\tau-2q+2)}{2\pi i} \iint_\Omega \frac{\lambda^{2-2q}(\zeta)\overline{\psi(\zeta)}\, d\zeta \wedge d\overline{\zeta}}{\zeta(\zeta-1)\cdots(\zeta-2q+2)(\zeta-\tau)} \\ &= \iint_{\Omega/G} c(\tau)\lambda^{2-2q}(\zeta)\overline{\psi(\zeta)}\, d\zeta \wedge d\overline{\zeta}. \end{aligned}$$

Comparison with (4.3.1) yields the theorem for even k. For k odd, we use the potential F that vanishes at ∞ and at $0, 1, \ldots, 2q-3$.

Corollary. *We have the equality*

$$\Theta g_\tau = h_A^q.$$

In the next section we generalize the above corollary to motions A other than $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

Remark. Rank 2 parabolic groups figure prominently in the fundamental work of Earle and Marden [EM] on compactifications for moduli spaces. The above theorem for $q = 2$ is connected to quasiconformal mappings and is clearly encountered in [EM]. It also appears in Nag's work on Teichmüller theory [N]. Nag's methods (limited to $q = 2$) rely on the study of deformations of complex structures on tori.

5.3. Let A and B generate an arbitrary rank 2 parabolic group G. For $a \in \hat{\mathbb{C}}$, $a \neq \alpha = \alpha(A) = \alpha(B)$, with $B(a) \in \mathbb{C}$, define

$$g_{A,B,a}(z) = \frac{1}{2\pi i} \frac{1}{z - B(a)} \prod_{j=0}^{2q-2} \frac{B(a) - A^j(a)}{z - A^j(a)}, \qquad z \in \hat{\mathbb{C}}.$$

A straightforward calculation using (1.2.1) shows that for $C \in PSL(2, \mathbb{C})$

$$g_{C\circ A\circ C^{-1}, C\circ B\circ C^{-1}, C(a)}(C(z))C'(z)^q = C'(B(a))^{q-1} g_{A,B,a}(z), \quad \text{all } z \in \hat{\mathbb{C}},$$

or (equivalently)

$$C_q^*(g_{C\circ A\circ C^{-1}, C\circ B\circ C^{-1}, C(a)}) = C'(B(a))^{q-1} g_{A,B,a}.$$

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$ with Im $\tau \neq 0$, $a = 0$, we have

$$g_{A,B,a} = g_\tau.$$

Thus choosing $C \in PSL(2, \mathbb{C})$ with

$$C(\alpha) = \infty, \qquad C(a) = 0, \qquad C(A(a)) = 1,$$

we see that

$$C \circ A \circ C^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C \circ B \circ C^{-1} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix},$$

for some $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau \neq 0$. Hence

$$C_q^*(g_{C(B(a))}) = C'(B(a))^{q-1} g_{A,B,a}.$$

Assume now that $\alpha \in \mathbb{C}$. Then the above formula yields

$$C_q^*(g_{\tau(B)/\tau(A)}) = \left(\frac{(1+\tau(B))(a-\alpha)^2}{-\tau(A)(a-\alpha)^2} \right)^{q-1} g_{A,B,a}. \tag{5.3.1}$$

For $\alpha = \infty$, the corresponding formula is

$$C_q^*(g_{\tau(B)/\tau(A)}) = \frac{1}{\tau(A)^{q-1}} g_{A,B,a}. \tag{5.3.2}$$

Using the well-known relation

$$\Theta_G \circ C_q^* = C_q^* \circ \Theta_{CGC^{-1}}, \tag{5.3.3}$$

we obtain the following

Theorem. *Let A and B generate a rank 2 parabolic group G with fixed point α. Then*

$$\Theta_G(g_{A,B,a}) = \left(\frac{1+\tau(B)(a-\alpha)^2}{-\tau(A)(a-\alpha)^2} \right)^{1-q} h_A^q \quad \textit{for } \alpha \in \mathbb{C},$$
$$= \tau(A)^{q-1} h_A^q \quad \textit{for } \alpha = \infty.$$

Proof. We apply the Poincaré operator Θ_G to both sides of (5.3.1), use (5.3.3) and the Corollary to Theorem 5.2 along with (1.2.3).

Remark. Similar formulae hold for the function

$$\frac{1}{2\pi i} \frac{1}{z - B(a)} \frac{B(a) - \alpha}{z - \alpha} \prod_{j=0}^{2q-3} \frac{B(a) - A^j(a)}{z - A^j(a)}.$$

References

[EM] C. J. Earle and A. Marden, *Geometric complex coordinates for Teichmüller space* (to appear).

[GK] F. P. Gardiner and I. Kra, *Stability of Kleinian groups*, Indiana Univ. Math. J. **21** (1972), 1037–1059.

[K1] I. Kra, *On cohomology of Kleinian groups. IV. The Ahlfors-Sullivan construction of holomorphic Eichler integrals*, J. Analyse Math. **43** (1983/84), 51–87.

[K2] ——, *On the vanishing of and spanning sets for Poincaré series for cusp forms*, Acta Math. **153** (1984), 47–116.

[K3] I. Kra, *Cusp forms associated to loxodromic elements of Kleinian groups*, Duke Math. J. **52** (1985), 587–625.

[N] S. Nag, *On some involutions and retractions arising in Teichmüller spaces*, J. Indian Math. Soc. **46** (1982), 1–29.

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