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AN EMBEDDING SPACE TRIPLE OF THE UNIT INTERVAL INTO A GRAPH AND ITS BUNDLE STRUCTURE

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ABSTRACT. Let l_2 denote a Hilbert space, and let

$$l_2^Q = \{(x_i) \in l_2 | \sup |i \cdot x_i| < \infty\} \text{ and}$$
$$l_2^f = \{(x_i) \in l_2 | x_i = 0 \text{ except for finitely many } i\}.$$

We show that the triple $(H(X), H^{\mathrm{LIP}}(X), H^{\mathrm{PL}}(X))$ of spaces of homeomorphisms, of Lipschitz homeomorphisms, and of PL homeomorphisms of a finite graph X onto itself is an (l_2, l_2^Q, l_2^f)-manifold triple, and that the triple $(E(I, X), E^{\mathrm{LIP}}(I, X), E^{\mathrm{PL}}(I, X))$ of spaces of embeddings, of Lipschitz embeddings, and of PL embeddings of $I = [0, 1]$ into a graph X is an (l_2, l_2^Q, l_2^4)-manifold triple.

Let l_2 denote a Hilbert space, and let

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An (l_2, l_2^Q, l_2^f)-*manifold triple* is defined as a triple (M, N, W) of an l_2-manifold, an l_2^Q-manifold, and an l_2^f-manifold which admits an open cover $\mathscr{U}$ of M and open embeddings $\varphi_U \colon U \to l_2$, $U \in \mathscr{U}$, such that $\varphi_U(U \cap N) = \varphi_U(U) \cap l_2^Q$ and $\varphi_U(U \cap W) = \varphi_U(U) \cap l_2^f$ [SW$_2$]. If X is a compact Euclidean polyhedron with $\dim X > 0$ and Y is an open set in $\mathbb{R}^n$, the triple $(C(X, Y), \mathrm{LIP}(X, Y), \mathrm{PL}(X, Y))$ of spaces of (continuous maps), of Lipschitz maps, and of PL maps of X into Y is such a manifold triple [Sa]. In this note, we find other examples of such manifold triples of function spaces where every function space has the compact-open topology; that is, we have the

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following:

Theorem 1. *For a finite graph (polyhedron of* $\dim = 1$ *)* $X \subset \mathbb{R}^n$, *the triple* $(H(X), H^{\mathrm{LIP}}(X), H^{\mathrm{PL}}(X))$ *of the spaces of homeomorphisms, of Lipschitz homeomorphisms and of PL homeomorphisms of* X *onto itself is an* (l_2, l_2^Q, l_2^f)-*manifold triple.*

Theorem 2. *For a graph* $X \subset \mathbb{R}^n$, *the triple* $(E(I, X), E^{\mathrm{LIP}}(I, X), E^{\mathrm{PL}}(I, X))$ *of spaces of embeddings, of Lipschitz embeddings, and of PL embeddings of* $I = [0, 1]$ *into* X *is an* (l_2, l_2^Q, l_2^f)-*manifold triple.*

By Theorem 1, Conjecture 2.6 in [SW$_1$] is true in the 1-dim case. Let $H_\partial(I) = \{h \in H(I) | h|\partial I = \mathrm{id}\}$, where $\partial I = \{0, 1\}$. Similarly, $H_\partial^{\mathrm{LIP}}(I)$ and $H_\partial^{\mathrm{PL}}(I)$ are defined. First we prove the following:

Theorem 3. $(H_\partial(I), H_\partial^{\mathrm{LIP}}(I), H_\partial^{\mathrm{PL}}(I))$ *is homeomorphic* $(\cong)$ *to* (l_2, l_2^Q, l_2^f).

Proof. For simplicity, let $H = H_\partial(I)$, $H' = H_\partial^{\mathrm{LIP}}(I)$, $H'' = H_\partial^{\mathrm{PL}}(I)$. For each $m \in \mathbb{N}$, let $L_m = \{h \in H' | \operatorname{bilip} h \leq 1 + m\}$, where $\operatorname{bilip} h$ is the minimum of $k \geq 1$ such that

$$k^{-1} \cdot |x - y| \leq |h(x) - h(y)| \leq k \cdot |x - y| \quad \text{for each } x, y \in I.$$

If $m < m'$, then L_m is a Z-set in $L_{m'}$. In fact, we have a homotopy $\varphi\colon L_{m'} \times I \to L_{m'}$ defined as follows:

$$\varphi_t(h)(s) = \left(1 - \frac{t}{2}\right) \cdot h\left(\frac{s}{1 - t/2}\right) \quad \text{for } 0 \leq s \leq 1 - \frac{t}{2},$$

$$\varphi_t(h)\left(1 - \frac{t}{4}\right) = 1 - \frac{(1 + m') \cdot t}{4},$$

and

$$\varphi_t(h) \text{ is linear on } \left[1 - \frac{t}{2}, 1 - \frac{t}{4}\right] \quad \text{and} \quad \text{on } \left[1 - \frac{t}{4}, 1\right],$$

which satisfies $\varphi_0 = \mathrm{id}$ and $\operatorname{Im}(\varphi_t) \cap L_m = \emptyset$ if $t > 0$. Since $H' = \bigcup_{m \in \mathbb{N}} L_m$ and each L_m is a compact convex set in the Banach space $C(I, \mathbb{R})$, which contains an infinite-dimensional, σ-fd-compact, convex set $H'' \cap L_m$ as a dense subset, $(L_m, H'' \cap L_m) \cong (Q, \sigma)$ by [Do, Theorem 2(i)]. Thus the tower $\{L_m\}_{m \in \mathbb{N}}$ satisfies the condition $(**)$ in [SW$_2$]. Let $\psi_n\colon H \to H''$, $n \in \mathbb{N}$, be maps such that $\psi_n(h)(\frac{i}{n}) = h(\frac{i}{n})$ and $\psi_n(h)$ is linear on each $[\frac{i-1}{n}, \frac{i}{n}]$. Then ψ_n converges to id as $n \to \infty$. For each $h \in H$, $\operatorname{bilip} \psi_n(h) = \max\{a, b^{-1}\}$, where

$$a = \max\{n \cdot (\psi_n(h)(\tfrac{i}{n}) - \psi_n(h)(\tfrac{i-1}{n})) | i = 1, \dots, n\},$$
$$b = \min\{n \cdot (\psi_n(h)(\tfrac{i}{n}) - \psi_n(h)(\tfrac{i-1}{n})) | i = 1, \dots, n\}.$$

Hence $\operatorname{bilip} \psi_n(h) \leq \operatorname{bilip} h$. It follows that H' is map dense in H and $\{L_m\}_{m \in \mathbb{N}}$ satisfies the condition $(*)'$ in [SW$_2$]. Since $H \cong l_2$ by [An] [cf.

Ke], $(H, H', H'') \cong (l_2, l_2^Q, l_2^f)$ by [SW_2, Lemma 1.5 and Theorems 2.1 and 2.2]. $\square$

By the arguments in [An], Theorem 1 follows from Theorem 3. Since [An] is unpublished, we give a sketch of the proof for the reader: Let A, B, and C be the sets of isolated points, of end points, and of branch points of X, respectively. Let D be the set of components of X that are simple closed curve. Let E and F be the sets of maximal free open arcs in $X \setminus \bigcup D$ such that the closure of each member of E is an arc and the closure of each member of F a simple closed curve. Let W be the finite set of all permutations of the union $A \cup B \cup C \cup D \cup E \cup F$ onto itself which carries each A, B, C, D, E, and F onto itself and preserves incidence in X, and let $n(W)$, $n(D)$, $n(E)$, and $n(F)$ denote the numbers of elements in W, D, E, and F, respectively. Let T be a finite space of $n(W) \cdot 2^{n(D)+n(F)}$ points, and let V be the product space of $n(D)$ circles except where $n(D) = 0$, in which case it is a single point. Then it is not hard to see that

$$(H(X), H^{\mathrm{LIP}}(X), H^{\mathrm{PL}}(X)) \cong (H \times T \times V, H' \times T \times V, H'' \times T \times V),$$

which implies Theorem 1 by Theorem 3. $\square$

We prove Theorem 2 in a more general setting. To this end, we extend the piecewise linearity to maps from I to a metric space $X = (X, d)$. A map $f: [a, b] \to X$ is said to be *linear* if

$$\frac{d(f(t), f(a))}{d(f(t), f(b))} = \frac{|t-a|}{|t-b|} \quad \text{for each } a < t < b,$$

and $f: [a, b] \to X$ is *piecewise linear* (PL) if there is a sequence $a = s_0 < s_1 < \cdots < s_n = b$ such that each $f|[s_{i-1}, s_i]$ is linear. The space of PL embeddings of I into a metric space X is also denoted by $E^{\mathrm{PL}}(I, X)$. In case X is a connected polyhedron in $\mathbb{R}^n$, we adopt the arc-length metric d defined by using Euclidean metric. Then the piecewise linearity of $f: I \to X$ defined above coincides with the usual sense. In this case, $E^{\mathrm{LIP}}(I, X)$ is not changed, since d is locally Lipschitz-equivalent to the Euclidean metric by [LV, Theorem 2.34]. Note that this metric d is *convex*; that is, for each $x, y \in X$ there is some $z \in X$ such that $d(x, z) = d(y, z) = d(x, y)/2$. Let $a(X)$ denote the hyperspace of arcs in X with the Vietoris topology (Hausdorff metric) and $\mathrm{Im}: E(I, X) \to a(X)$ the natural map defined by $\mathrm{Im}(h) = h(I)$. Theorem 2 is a corollary to the following:

Theorem 4. *Let X be a locally compact* 1-dim ANR X *with a metric d which is convex on a neighborhood of each point. Then $(E(I, X), E^{\mathrm{LIP}}(I, X), E^{\mathrm{PL}}(X, Y))$ is an (l_2, l_2^Q, l_2^f)-manifold triple and the map* $\mathrm{Im}: E(I, X) \to a(X)$ *is a locally trivial bundle with fiber $\mathbb{Z}_2 \times l_2$ and* $\mathrm{Im}|E^{\mathrm{LIP}}(I, X)$ *and* $\mathrm{Im}|E^{\mathrm{PL}}(I, X)$ *are subbundles with fibers $\mathbb{Z}_2 \times l_2^Q$ and $\mathbb{Z}_2 \times l_2^f$, respectively.*

Before the proof, note that any connected locally compact 1-dim ANR has a convex metric. In fact, it has a Peano compactification with locally nonseparating remainder by [Cu] and any Peano continuum admits a convex metric by [Bi] or [Mo].

In Theorem 4, for each $h \in E(I, X)$, there is a dendrite (= compact 1-dim AR) Y such that $h(I) \subset \operatorname{int} Y$, whence $E(I, Y)$ is a neighborhood of h in $E(I, X)$. The arc-length metric of Y defined by using the metric of X is convex and locally coincides with the metric of X. Thus Theorem 4 reduces to the case in which X is a dendrite with a convex metric.

Theorem 5. *For a dendrite X with a convex metric, $(E(I, X), E^{\mathrm{LIP}}(I, X), E^{\mathrm{PL}}(I, X))$ is an (l_2, l_2^Q, l_2^f)-manifold triple and the map $\mathrm{Im}\colon E(I, X) \to a(X)$ is a locally trivial bundle with fiber $\mathbb{Z}_2 \times l_2$ and $\mathrm{Im}\,|E^{\mathrm{LIP}}(I, X)$ and $\mathrm{Im}\,|E^{\mathrm{PL}}(I, X)$ are subbundles with fibers $\mathbb{Z}_2 \times l_2^Q$ and $\mathbb{Z}_2 \times l_2^f$, respectively.*

Proof. For simplicity, let $E = E(I, X)$, $E' = E^{\mathrm{LIP}}(I, X)$ and $E'' = E^{\mathrm{PL}}(I, X)$ and let H, H' and H'' be as in the proof of Theorem 1.1. Let $b(X) = X^2 \backslash \Delta X$, where ΔX is the diagonal of X^2, and let $\beta\colon E \to b(X)$ be the map defined by $\beta(h) = (h(0), h(1))$. We have the map $\alpha\colon b(X) \to a(X)$ such that $\alpha(x, y)$ is the unique arc in X connecting x and y. Then $\alpha \circ \beta = \mathrm{Im}\colon E(I, X) \to a(X)$, and α is a locally trivial bundle with fiber $\mathbb{Z}_2$. (Geometrically, $b(X)$ can be considered as the space of oriented arcs in X.) Hence it suffices to construct a homeomorphism

$$\varphi\colon (E, E', E'') \to (b(X) \times H, b(X) \times H', b(X) \times H'')$$

so that $p \circ \varphi = \beta$, where $p\colon b(X) \times H \to b(X)$ is the projection. Then (E, E', E'') is an (l_2, l_2^Q, l_2^f)-manifold triple by Theorem 3 and the result of [SW_2]. From the uniquely arcwise connectedness of X, there exists a map $\lambda\colon X^2 \times I \to X$ such that

$$d(x, \lambda(x, y, t)) = t \cdot d(x, y) \quad \text{and} \quad d(y, \lambda(x, y, t)) = (1 - t) \cdot d(x, y)$$

for each $x, y \in X$ and $t \in I$. We define the map $\tau\colon b(X) \to E$ by $\tau(x, y)(t) = \lambda(x, y, t)$. As is easily observed, $\tau(b(X)) \subset E''$. From the uniquely arcwise connectedness of X, $\tau \circ \beta(h)(I) = h(I)$ for each $h \in E$. Then the desired homeomorphism φ and its inverse are defined by

$$\varphi(h) = (\beta(h), (\tau \circ \beta(h))^{-1} \circ h) \quad \text{and} \quad \varphi^{-1}(x, y, g) = \tau(x, y) \circ g. \quad \square$$

Example. If X contains a two-disk, both $\mathrm{Im}\colon E(I, X) \to a(X)$ and $\mathrm{Im}\colon E^{\mathrm{LIP}}(I, X) \to a(X)$ are not locally trivial bundles. To show this, let $g\colon I^2 \to X$ be an embedding and $A_0 = g(I \times \{0\})$. For each $n \in \mathbb{N}$, let

$$A_n = g\left(I \times \left\{\frac{1}{2n}, \frac{1}{2n-1}\right\} \cup \{1\} \times \left[\frac{1}{2n}, \frac{1}{2n-1}\right]\right) \in a(X).$$

Then A_n converges to A_0 in $a(X)$. However, any $h_n \in \mathrm{Im}^{-1}(A_n)$ $(n \in \mathbb{N})$ does not converge to any $h \in \mathrm{Im}^{-1}(A_0)$, because $\{h_n(0), h_n(1)\}$ converges to

$\{g(0,0)\}$, but $\{h(0), h(1)\} = \{g(0,0), g(1,0)\}$. In case X is a polyhedron with $\dim X > 1$, $\mathrm{Im}\colon E^{\mathrm{PL}}(I, X) \to a^{\mathrm{Pol}}(X)$ is not a locally trivial bundle, where $a^{\mathrm{Pol}}(X)$ is the subspace of $a(X)$ consisting of polyhedral arcs.

Problem. Let X and Y be Euclidean polyhedra such that X is compact and $E(X, Y) \neq \varnothing$. Is $(E(X, Y), E^{\mathrm{LIP}}(X, Y), E^{\mathrm{PL}}(X, Y))$ an (l_2, l_2^Q, l_2^f)-manifold triple? Is each space of this triple an ANR? If $X = I$ and $\dim Y > 1$, is it then an ANR?

Remark. In Theorem 5, $\mathrm{Im}\colon E(I, X) \to a(X)$ is nontrivial in the case $X \not\cong I$. In fact, $\mathrm{Im} = \alpha \circ \beta$ and β is trivial as shown in the proof, but α is nontrivial since $b(X) = X^2 \backslash \Delta X$ is connected in this case. It should be noted that $a(X)$ is not contractible in general. For example, in the case X is the simple triod, $a(X)$ has the homotopy type of S^1.

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