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COMMENTS ON AN L^2 INEQUALITY OF A. K. VARMA INVOLVING THE FIRST DERIVATIVE OF POLYNOMIALS

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ABSTRACT. Let t_n be a trigonometric polynomial of degree n with real coefficients, and let $w(x) \in C^2[0, \pi]$ be nonnegative. Employing a well-known result of G. Szegő, we study the extremal property of the integral

$$\int_0^\pi (t_n'(x))^2 w(x)\,dx,$$

subject to the constraint $\|t_n\|_\infty \le 1$.

1. INTRODUCTION

Let $w(x)$ be a nonnegative function defined on the interval $[0, \pi]$ and $t_n(x)$ be a trigonometric polynomial of degree n with real coefficients such that $\|t_n\|_\infty = \max_{0\le x\le 2\pi} |t_n(x)| \le 1$. The intention of this paper is to study the following quantity:

$$\sup_{\|t_n\|_\infty \le 1} \int_0^\pi (t_n'(x))^2 w(x)\,dx.$$

Recently, A. K. Varma [2] considered the above problem for $w(x) = \sin x$ and proved the following:

Theorem A. *Let t_n be a trigonometric polynomial of degree n with real coefficients such that $\|t_n\|_\infty \le 1$. Then*

$$\int_0^\pi (t_n'(x))^2 \sin x\,dx \le n^2\left(1 + \frac{1}{4n^2 - 1}\right) = \int_0^\pi (C_n'(x))^2 \sin x\,dx,$$

where $C_n(x) = \cos nx$.

Exploring Varma's method, we will establish, among other things, the following theorem.

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Theorem 1. *Let* t_n *be the same as above. Then, for* $n \geq 2$,

$$\int_0^\pi (t_n'(x))^2 \sin^2 x\, dx \leq \int_0^\pi (C_n'(x))^2 \sin^2 x\, dx.$$

In §2, we will first establish a key identity and then derive Theorem 1 from it. A closely related problem of finding the extremal value for

$$\int_{-1}^1 (P_n'(x))^2 (1-x^2)^\alpha\, dx,$$

where P_n is a polynomial of degree n with real coefficients subject to the condition $\max_{|x|\leq 1} |P_n(x)| \leq 1$, and α is some suitably chosen real value, is discussed in §3.

2. A basic identity and its consequences

Our method is based on the following well-known result of G. Szegő:

Theorem B. *Let* t_n *be a trigonometric polynomial of degree* n *with real coefficients and* $|t_n(x)| \leq M$ *for all real* x. *Then*

$$n^2 t_n^2 + (t_n')^2 \leq M^2 n^2,$$

and the equality holds if and only if $t_n(x) = M\cos(nx+a)$ *for some real constant* a.

Imitating Varma's approach, we deduce the following:

Lemma 1. *Let* $w(x), f(x) \in C^2[a, b]$ *and* s *be any nonzero real number. Then*

$$\begin{aligned} &\int_a^b (f'(x))^2 \left(2w(x) + \frac{1}{2s^2} w''(x)\right) dx \\ &\quad = \frac{1}{2}\int_a^b [(f'(x))^2 + s^2 f^2(x)] \left(w(x) + \frac{1}{s^2} w''(x)\right) dx \\ &\qquad + \frac{1}{2}\int_a^b [s^2 (f'(x)/s)^2 + (f''(x)/s)^2] w(x)\, dx \\ &\qquad - \frac{1}{2s^2}\int_a^b (s^2 f(x) + f''(x))^2 w(x)\, dx + A + B, \end{aligned} \tag{2.1}$$

where

$$A = f(b)f'(b)w(b) - f(a)f'(a)w(a)$$

and

$$B = \tfrac{1}{2}(f^2(a)w'(a) - f^2(b)w'(b)).$$

Since this identity can be verified easily by multiplying out and combining the terms on each side, we omit the proof. We now deduce Theorem A from this lemma. To do so we choose $f(x) = t_n$ and $s = n$. Then, from Theorem B, we have

$$(t_n'(x))^2 + n^2 t_n^2(x) \leq n^2 \|t_n\|_\infty^2 \tag{2.2}$$

and

$$n^2(t_n'(x)/n)^2 + (t_n''(x)/n)^2 \le n^2\|t_n\|_\infty^2 .$$

We now assume that $w(x)$ satisfies the following properties on the interval $[0, \pi]$:

$$w(x) \ge 0, \quad w(0) = w(\pi) = 0 \tag{2.3}$$

and

$$w(x) + \frac{1}{n^2}w''(x) \ge 0 \quad \text{for every } n. \tag{2.4}$$

From (2.3), we see that, if $\|t_n\|_\infty \le 1$,

$$w'(0) \ge 0, \; w'(\pi) \le 0, \; A = 0 \quad \text{and} \quad B \le \tfrac{1}{2}(w'(0) - w'(\pi)). \tag{2.5}$$

From Lemma 1 and (2.2)–(2.5), we obtain

$$\int_0^\pi (t_n')^2 \left(w + \frac{1}{4n^2}w''\right) dx \le \frac{n^2}{2}W, \tag{2.6}$$

where $W = \int_0^\pi w(x)\,dx$; the equality holds if and only if $t_n = \pm\cos nx$. We now see that Theorem A follows from (2.6) by observing that $w(x) = \sin x$ satisfies all the required properties.

For a more general $w(x)$, we have the following less precise result. Since the proof is very straightforward, we omit it here.

Theorem 2. *Let $w \in C^2[0, \pi]$, $w \ge 0$, and $w(0) = w(\pi) = 0$, and let t_n be a trigonometric polynomial of degree n with real coefficients. Then*

$$\sup_{\|t_n\|_\infty \le 1} \int_0^\pi (t_n'(x))^2 w(x)\,dx = \frac{n^2}{2}W + E_n, \tag{2.7}$$

where E_n is a constant and

$$|E_n| \le \frac{1}{4}(w'(0) - w'(\pi)) + \frac{1}{2}\int_0^\pi |w''(t)|\,dt.$$

This theorem is sharp in the sense that, asymptotically, the supremum of the integral in (2.7) is almost achieved by choosing $t_n = \cos nx$.

We now prove Theorem 1. We will base our proof on the following lemma:

Lemma 2. *Let $w \in C^2[0, \pi]$ and t_n be the same as in Theorem A. If for each positive integer n, there exists a function $w_n \in C^2[0, \pi]$ satisfying the differential equation*

$$w_n + \frac{1}{4n^2}w_n'' = w \tag{2.8}$$

and the properties (2.3) and (2.4), then

$$\int_0^\pi (t_n'(x))^2 w(x)\,dx \le n^2 W_n/2,$$

where $W_n = \int_0^\pi w_n(x)\,dx$ and the equality holds if $t_n = \cos nx$.

The proof follows immediately from Lemma 1.

To prove Theorem 1, we choose $w = \sin^2 x$. It is easy to verify that

$$w_n = \frac{n^2}{n^2 - 1}\left(\sin^2 x - \frac{\sin^2 nx}{n^2}\right)$$

is a solution of the differential equation (2.8) and that it satisfies (2.3). Since

$$w_n + \frac{1}{n^2} w'' = (n^2 - 1)^{-1}(3\sin^2 nx + (n^2 - 4)\sin^2 x) \geq 0$$

for $n \geq 2$, we see that (2.4) is also satisfied. Theorem 2 now follows from Lemma 2.

It is worthwhile to point out that the above method fails for $w = \sin^3 x$, that is, for each integer n, the differential equation $w + w''/4n^2 = \sin^3 x$ does not possess a solution w_n satisfying conditions (2.3) and (2.4). To see this, we note that $w(x) = \sin^3 x$ satisfies the equation

$$w + \frac{1}{4n^2} w'' = \left(1 - \frac{9}{4n^2}\right) w + \frac{3}{2n^2}\sin x, \tag{2.9}$$

and $u = -6\sin x/(4n^2 - 1)$ satisfies the equation

$$w + \frac{1}{4n^2} w'' = -\frac{3}{2n^2}\sin x. \tag{2.10}$$

From (2.9) and (2.10), we see that the general solution of this differential equation satisfying (2.3) has the form

$$w_n = \left(1 - \frac{9}{4n^2}\right)^{-1}\left(\sin^3 x - \frac{6}{4n^2 - 1}\sin x\right) + c_n \sin 2nx. \tag{2.11}$$

Then

$$\lim_{x \to 0} \frac{w_n(x)}{\sin x} = -\frac{6}{4n^2 - 1}\left(1 - \frac{9}{4n^2}\right)^{-1} + 2nc_n \tag{2.12}$$

and

$$\lim_{x \to \pi} \frac{w_n(x)}{\sin x} = -\frac{6}{4n^2 - 1}\left(1 - \frac{9}{4n^2}\right)^{-1} - 2nc_n. \tag{2.13}$$

From (2.12) and (2.13), we see that it is impossible to choose c_n so that $w_n \geq 0$ in $[0, \pi]$.

3. A related problem

Let P_n be a polynomial of degree n with real coefficients and

$$\max_{x \in [-1, 1]} |P_n(x)| \leq 1.$$

Let $t_n(x) = P_n(\cos x)$. Then, from Theorems A and 1, we conclude that

$$\int_{-1}^{1} (P_n'(x))^2 (1-x^2)^\alpha \, dx \le \int_{-1}^{1} (T_n'(x))^2 (1-x^2)^\alpha \, dx \tag{3.1}$$

for $\alpha = 1$, $3/2$, where $T_n(x)$ is the Tchebycheff polynomial of degree n.

By choosing $w(x) = 1$ in Lemma 1 and observing that $t_n'(x) = -\sin x \, P_n'(\cos x) = 0$ for $x = 0$ and π, we conclude that (3.1) is also valid for $\alpha = 1/2$. It is also noted by Varma in [2] that, using the Gauss quadrature formula, (3.1) holds for $\alpha = -1/2$. And, employing a very different method, D. B. Bojanov [1] proved that (3.1) holds for $\alpha = 0$. In fact, he established the following theorem:

Theorem C. *Let P_n be a polynomial of degree n with real coefficients and $\max_{|x|\le 1} |P_n(x)| \le 1$. Then, for $1 \le p < \infty$,*

$$\int_{-1}^{1} |P_n'(x)|^p \, dx \le \int_{-1}^{1} |T_n'(x)|^p \, dx,$$

where T_n is the Tchebycheff polynomial of degree n.

And it will be interesting to know whether the L^2 norm in (3.1) can be extended to the L^p norm so that a result similar to Theorem C can be obtained for different values of α.

References

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