

THE CLASSIFICATION OF SOLENOIDS

J. M. AARTS AND R. J. FOKKINK

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ABSTRACT. An elementary proof is presented of a classification theorem for solenoids.

1. INTRODUCTION

A *solenoid* may be defined as the intersection of a sequence of tori T_1, T_2, \dots such that T_{i+1} is wrapped around inside T_i longitudinally n_i times in a smooth fashion without folding back. There is no loss of generality in assuming that all n_i are prime numbers greater than one. A solenoid is denoted by S_P , where $P = (p_1, p_2, \dots)$ is a sequence of prime numbers.

We call the sequences P and Q of primes *equivalent* and write $P \sim Q$ if a finite number of primes can be deleted from each sequence so that every prime number occurs the same number of times in the deleted sequences. The following classification theorem was conjectured by Bing [3], who noticed the correctness of the “if” part.

Theorem. *The solenoids S_P and S_Q are homeomorphic iff $P \sim Q$.*

The “only if” part was proved by McCord [5]. The proof exploits the fact that a solenoid can be endowed with the structure of a topological group. The continuity of the Čech cohomology and Pontrjagin duality are invoked. The aim of this paper is to present a rather elementary proof of the theorem.

2. MAPPING TORUS

In [6, p. 392] a quite elaborate description of the flow structure of a solenoid is presented. With the above definition of a solenoid, its flow structure may be captured as follows. The transverse (or cross) section of a solenoid is a Cantor set C . By following the arc components (or composants) in one of the two possible directions, in a natural way there is defined a return map $\sigma : C \rightarrow C$. The map σ essentially is the Poincaré return map. The solenoid then may be seen as the *mapping torus* (or suspension) $\Sigma(C, \sigma)$, which is obtained from the product $C \times [0, 1]$ by identifying each $(x, 1)$ with $(\sigma(x), 0)$, $x \in C$. By

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regarding $t \in [0, 1]$ as time, a transparent picture of the flow structure of the solenoid emerges.

We might as well have started out by *defining*, for a sequence of primes $P = (p_1, p_2, \dots)$, the solenoid S_P as the mapping torus $\sum(C_P, \sigma)$. Here C_P is the Cantor set represented as the topological product $C_P = \prod_{i=1}^{\infty} \{0, 1, \dots, p_i - 1\}$ of the finite discrete spaces $\{0, 1, \dots, p_i - 1\}$, $i = 1, 2, \dots$.

The return map σ has the form

$$\begin{aligned} \sigma(x_1, x_2, \dots) &= (x_1 + 1, x_2, x_3, \dots) \quad \text{if } x_1 < p_1 - 1, \\ \sigma(p_1 - 1, \dots, p_{k-1} - 1, x_k, \dots) &= (0, 0, \dots, 0, x_k + 1, \dots) \quad \text{if } x_k < p_k - 1, \\ \sigma(p_1 - 1, p_2 - 1, \dots) &= (0, 0, \dots). \end{aligned}$$

For convenience we shall write $\bar{p} = \{0, 1, \dots, p - 1\}$ for any number $p \geq 1$. Thus $C_P = \prod_{i=1}^{\infty} \bar{p}_i$. The cascade (C_P, σ) is sometimes called an *adding machine*. Actually C_P is the topological group of P -adic integers, and the return map corresponds to the addition of $(1, 0, 0, \dots)$ [4].

The representation of the solenoid $S_P = \sum(C_P, \sigma)$ as a mapping torus is by no means unique. If U is any clopen subset of C_P , then the return map σ_U can be defined in the same way as above. The map $\sigma_U : U \rightarrow U$ can be seen to be a homeomorphism and we find $S_P = \sum(U, \sigma_U)$.

There is a simple relation between σ and σ_U . For each $x \in U$, let $n(x)$ be the least integer ≥ 1 such that $\sigma^{n(x)}(x) \in U$. Then $\sigma_U(x) = \sigma^{n(x)}(x)$. This motivates the following definition.

Definition. Let $P = (p_1, p_2, \dots)$ and $Q = (q_1, q_2, \dots)$ be sequences of primes. The adding machines (C_P, σ) and (C_Q, σ) are *first return equivalent* if there exist clopen subsets U and V of C_P and C_Q , respectively, such that (U, σ_U) and (V, σ_V) are conjugate, that is, there is a homeomorphism $h : U \rightarrow V$ such that $h \circ \sigma_U = \sigma_V \circ h$.

3. THE CLASSIFICATION THEOREM

Theorem. Let $P = (p_1, p_2, \dots)$ and $Q = (q_1, q_2, \dots)$ be sequences of primes. The following conditions are equivalent.

- (a) The spaces S_P and S_Q are homeomorphic.
- (b) The adding machines (C_P, σ) and (C_Q, σ) are first return equivalent.
- (c) $P \sim Q$.

Proof. (a) and (b) are equivalent. This is essentially proved in [7] and a similar proof can be found in [1] and [2].

(b) and (c) are equivalent. First suppose that $P \sim Q$. We may assume that the primes to be deleted are p_1, \dots, p_s and q_1, \dots, q_t , since a change in the order of any finite sequence of primes does not change the isomorphism type of the adding machine. Let $P' = (p_{s+1}, p_{s+2}, \dots)$ and $Q' = (q_{t+1}, q_{t+2}, \dots)$ be the derived sequences. We claim that the return maps to $U = \{0\} \times \dots \times \{0\} \times \prod_{i=s+1}^{\infty} \bar{p}_i$ and $V = \{0\} \times \dots \times \{0\} \times \prod_{i=t+1}^{\infty} \bar{q}_i$ are conjugate. In other words,

$(C_{P'}, \sigma)$ and $(C_{Q'}, \sigma)$ are conjugate. At first the conjugating homeomorphism is defined on one orbit only: let $e = (0, 0, 0, \dots)$; then $h(\sigma^n(e)) = \sigma^n(e)$. The map h is an isomorphism from \mathbb{Z} with the P' -adic topology to \mathbb{Z} with the Q' -adic topology. These subgroups are dense in $C_{P'}$ and $C_{Q'}$, respectively. Therefore h can be extended to a conjugating isomorphism.

To prove (b) \rightarrow (c), let U and V be clopen subsets of (C_P, σ) and (C_Q, σ) , respectively, and let $h : U \rightarrow V$ be a homeomorphism, conjugating σ_U and σ_V . Let $x \in U$ and write $y = h(x)$. Without loss of generality we may assume that $U = \{x_1\} \times \dots \times \{x_k\} \times \prod_{i=k+1}^{\infty} \bar{p}_i$. Because of compactness there exists a number l and there are points y^1, \dots, y^N such that V is the disjoint union of the sets $\{y_1^j\} \times \dots \times \{y_l^j\} \times \prod_{i=l+1}^{\infty} \bar{q}_i$, $j = 1, \dots, N$. It is to be observed that these N sets are permuted by σ_V . Let m be any number greater than $l + 1$. By continuity of h there is an n such that

$$h\left(\{x_1\} \times \dots \times \{x_n\} \times \prod_{i=n+1}^{\infty} \bar{p}_i\right) \subset \{y_1\} \times \dots \times \{y_m\} \times \prod_{i=m+1}^{\infty} \bar{q}_i.$$

Inspection of the adding machines and their conjugation shows that for some number M we have

$$p_{k+1} \cdot p_{k+2} \cdot \dots \cdot p_n = M \cdot N \cdot q_{l+1} \cdot \dots \cdot q_m.$$

Indeed an element of U returns to $\{x_1\} \times \dots \times \{x_n\} \times \prod_{i=n+1}^{\infty} \bar{p}_i$ precisely once every $p_{k+1} \cdot p_{k+2} \cdot \dots \cdot p_n$ times, whereas an element of V returns to $\{y_1\} \times \dots \times \{y_m\} \times \prod_{i=m+1}^{\infty} \bar{q}_i$ precisely once every $N \cdot q_{l+1} \cdot \dots \cdot q_m$ times. We conclude that $p_{k+1} \cdot \dots \cdot p_n$ must be divisible by $N \cdot q_{l+1} \cdot \dots \cdot q_m$. This shows that the tail $(q_{l+1}, q_{l+2}, \dots)$ of Q can be mapped injectively in the sequence P . Reversing the roles of P and Q we find that some tail of P can be mapped injectively in Q . It then easily follows that $P \sim Q$.

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