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$\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$ REQUIRES AN INACCESSIBLE

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ABSTRACT. We show that if there is a simplified $(\omega_2, 1)$-morass with linear limits and $2^{\aleph_1} = \aleph_2$, then $\omega_3\omega_1 \nrightarrow (\omega_3\omega_1, 3)^2$. Thus, assuming $2^{\aleph_1} = \aleph_2$, this negative relation holds in V if both $\aleph_2$ and $\aleph_3$ are (successor cardinals)^L, since in this case, well-known arguments show there is a simplified $(\omega_2, 1)$-morass with linear limits. The contrapositive is that, assuming $2^{\aleph_1} = \aleph_2$, the positive relation holds only if either $\aleph_2$ or $\aleph_3$ is (inaccessible)^L.

INTRODUCTION

This paper builds on the work of T. Miyamoto's dissertation [M]. After the appearance of [SS1], several people, including Stanley and Velleman, conjectured that there should be a morass construction of a partition verifying $\omega_3\omega_1 \nrightarrow (\omega_3\omega_1, 3)^2$. Miyamoto proved this conjecture in [M, pp. 67–94] by splitting the forcing of §2 of [SS1] into two stages. The first can be viewed as generically adding a simplified $(\omega_2, 1)$-morass with some extra structure. The second stage used *any such* morass (not necessarily a generic one) to construct a partition as above.

This paper is concerned with the second stage. Miyamoto's extra structure did not seem to fit nicely into a catalogue of properties developed in [V2, V3]. At a meeting in Oberwolfach in July 1988, while trying to understand the proof in [M], Stanley and Velleman found the proof presented in this paper. Later, they learned that Morgan had independently found essentially the same proof. Miyamoto uses something quite similar to linear limits, but our main improvement over [M] is that we have been able to replace the rest of his extra structure by the better-understood (and apparently weaker, see below, (1.3)) complete amalgamation system, whose existence, for any simplified $(\omega_2, 1)$-morass, follows from $2^{\aleph_1} = \aleph_2$, viz. [V3]. If $\kappa = \lambda^+$ and $A \subseteq \kappa$ and $\kappa = (\lambda^+)^{L[A\cap\lambda]}$, then it is clear that the construction in [D] of a simplified $(\kappa, 1)$-morass with linear limits can be carried out in $L[A]$. This holds, in particular for $\kappa = \aleph_2$.

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If $\aleph_3^{L[A]} = \aleph_3$, then the $L[A]$ $(\omega_2, 1)$-morass with linear limits is an $(\omega_2, 1)$-morass with linear limits in V. If both $\aleph_2$ and $\aleph_3$ are (successor cardinals)^L, then such an A can be found. As mentioned above if $2^{\aleph_1} = \aleph_2$ then this morass (or any other one) has a complete analgamation system. Summarizing:

Proposition. *If* $2^{\aleph_1} = \aleph_2$ *and there is a simplified* $(\omega_2, 1)$*-morass with linear limits then* $\omega_3\omega_1 \nrightarrow (\omega_3\omega_1, 3)^2$.

Theorem. *If* $2^{\aleph_1} = \aleph_2$ *and* $\omega_3\omega_1 \rightarrow (\omega_3\omega_1, 3)^2$ *then either* $\aleph_2$ *or* $\aleph_3$ *is (inaccessible)*^{L}.

Corollary. $\mathrm{Con}(ZFC + 2^{\aleph_1} = \aleph_2 + \omega_3\omega_1 \rightarrow (\omega_3\omega_1, 3)^2) \Rightarrow \mathrm{Con}(ZFC + \exists$ *an inaccessible cardinal*).

In [SS1, §5], it was shown that Con(ZFC + $\exists$ a weakly compact cardinal) $\Rightarrow$ Con(ZFC + $2^{\aleph_1} = \aleph_2 + \forall k < \omega(\omega_3\omega_1 \rightarrow (\omega_3\omega_1, k)^2)$). In the extension, $2^{\aleph_0} = \min((2^{\aleph_0})^V, \aleph_2)$; full GCH can be obtained or violated in the usual ways, as desired. However, nothing is known about the consistency of $2^{\aleph_1} > \aleph_2 + \omega_3\omega_1 \rightarrow (\omega_3\omega_1, 3)^2$. See [SS2] for a discussion of the difficulties here.

In §2 we prove the proposition. In §1 we provide further background, including a sketch of notation and terminology, and of the parallels between §2 of [SS1] and our adaptation (and in one instance, improvement) of that material to the morass setting. See [SS1] for a more thorough discussion of the historical background of work on $\omega_3\omega_1 \rightarrow (\omega_3\omega_1, 3)^2$. While we will sometimes discuss the relationship between our proof and the proofs in [SS1, M] for the benefit of readers familiar with those papers, familiarity with [SS1] or [M] is not necessary to follow our proof.

1. Preliminaries

(1.1) **Partitions.** All unexplained notation is standard. Though $\omega_1, \omega_2, \omega_3$ can be replaced throughout by $\kappa, \kappa^+, \kappa^{++}$, κ regular and uncountable, for concreteness we stick to $\omega_1, \omega_2, \omega_3$. We shall only explain the single instance of the ordinary partition symbol which we actually use. See [EHMR] for a comprehensive treatment.

Definition. $\omega_3\omega_1 \rightarrow (\omega_3\omega_1, 3)^2$ means: Given any disjoint partition of the two element subsets of the ordinal $\omega_3\omega_1$ (ordinal multiplication), $[\omega_3\omega_1]^2 = C_1 \cup C_2$, either there is a subset of $\omega_3\omega_1$ of order type $\omega_3\omega_1$, all of whose two element subsets lie in C_1 (a "large homogeneous red set") or there is a three-element subset of $\omega_3\omega_1$ whose three two-element subsets lie in C_2 (a "green triangle"). Replacing $\rightarrow$ by $\nrightarrow$ negates the preceding statement.

The "coloring" metaphor is traditional, since a disjoint partition of $[X]^2$ can be thought of as coloring the edges of the complete graph on X by two colors: red for the edges of C_1 and green (rather than the traditional blue of [EHMR]) for the edges in C_2. We shall use this colorful imagery in what follows. We view $\omega_3\omega_1$ as $\omega_1 \times \omega_3$, ordered lexicographically; thus, the ordinal $\omega_3\gamma + \rho$ is identified with (γ, ρ). For $x = (\gamma, \rho) \in \omega_1 \times \omega_3$, we write $\gamma(x) = \gamma$, $\rho(x) = \rho$; the former is the column index of x, the latter the row index of x. A horizontal edge is an edge $\{x, y\}$ with $\rho(x) = \rho(y)$; similarly for vertical and γ. A descending edge is an edge $\{x, y\}$ where, e.g., $\gamma(x) < \gamma(y)$ and $\rho(x) > \rho(y)$; similarly for ascending and $\rho(x) < \rho(y)$.

It is not too hard to show that if there is a coloring verifying $\omega_3\omega_1 \nrightarrow (\omega_3\omega_1, 3)^2$ then there is one which satisfies (1) below; this is proved in [SS1] for the weaker $(1')$:

(1) the only green edges are descending edges.
$(1')$ no horizontal nor vertical edges are green.

To prove (1), we start from a coloring verifying $\omega_3\omega_1 \nrightarrow (\omega_3\omega_1, 3)^2$ and satisfying $(1')$ and we find a subset of the vertex set of order type $\omega_3\omega_1$ (ω_3 "high" on $\aleph_1$ "columns") devoid of ascending green edges. For this it is enough to note that for any vertex x, $\{y : \{x, y\}$ is green$\}$ is homogeneous red, lest there be a green triangle. In particular, for no vertex x does $\{y : \{x, y\}$ is green$\}$ have order type $\omega_3\omega_1$. Thus, for all vertices x there is $w(x)$ such that $\gamma(x) \le \gamma(w(x))$, $\rho(x) \le \rho(w(x))$ and if $\gamma(x) < \gamma(y)$ and $\{x, y\}$ is ascending and green then either $\gamma(y) \le \gamma(w(x))$ or $\rho(y) \le \rho(w(x))$. It is then easy to recursively construct the desired subset of order type $\omega_3\omega_1$: having added a vertex x, add no y "above and to the right of" x unless it is also "above and to the right of" $w(x)$.

These observations allow us to color fewer edges. We shall not color vertical edges and we shall view the remaining edges as *ordered pairs* where the vertex with smaller γ comes first; thus, for $H \subseteq \omega_1 \times \omega_3$ we let $(H)^2 = \{\langle x, y\rangle \in H \times H : \gamma(x) < \gamma(y)\}$. We shall color $(\omega_1 \times \omega_3)^2$. We are apparently wasting paint by coloring horizontal and ascending edges, since we know that, WLOG, they are not green. This has to do with our decision to follow [SS1] in using a third color, gray. If $\gamma(x) < \gamma(y) < \gamma(z)$, $\langle x, y\rangle$ and $\langle x, z\rangle$ are both green then, as above in our discussion of (1), $\langle y, z\rangle$ *cannot* be green, lest there be a green triangle. As in the discussion of (1), *too many* green edges can impose a large homogeneous red set, but there must be *enough* green edges to prevent the complement from being large.

The solution in [SS1] is to reserve red for edges which *cannot* be green lest there be a green triangle. We will choose to color some descending edges green. Edges which are not required to be red and which we did not choose to color green (perhaps because they are not descending) will be colored gray. The gray edges are "potentially green" in amalgamations, where we consider the descending "mixed" edge consisting of one of the original vertices and the twin of the other (see below, following (6) of (2.2) in the proof of the proposition). Coloring edges gray, including horizontal and ascending ones, preserves them as a stock of potential greens. When the construction is complete, the grays will be recolored red.

(1.2) **Morasses.** We briefly review here the definitions of simplified morasses and associated structures which we will be using. For a more complete treatment the reader should see [V1, V2, or V3]. Throughout this section κ will stand for a regular uncountable cardinal. Suppose $\langle\theta_\alpha : \alpha \le \kappa\rangle$ is a sequence of ordinals such that $\forall\alpha < \kappa(0 < \theta_\alpha < \kappa)$ and $\theta_\kappa = \kappa^+$, and for each $\alpha < \beta \le \kappa$, $\mathscr{F}_{\alpha\beta}$ is a set of order preserving functions from θ_α to θ_β.

Definition. The structure $\langle\langle\theta_\alpha : \alpha \le \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \le \kappa\rangle\rangle$ is called a *simplified* $(\kappa, 1)$-*morass* if the following requirements are satisfied.

(1) If $\alpha < \beta < \kappa$ then $|\mathscr{F}_{\alpha\beta}| < \kappa$.

(2) If $\alpha < \beta < \gamma \le \kappa$ then $\mathscr{F}_{\alpha\gamma} = \{f \circ g : f \in \mathscr{F}_{\beta\gamma}, g \in \mathscr{F}_{\alpha\beta}\}$.

(3) For all $\alpha < \kappa$, $\mathscr{F}_{\alpha,\alpha+1}$ is either a singleton or a pair $\{f, g\}$ such that for some $\sigma < \theta_\alpha$, $f \restriction \sigma = g \restriction \sigma$ and $f''\theta_\alpha \subseteq g(\sigma)$. Such a pair is called an *amalgamation pair.*

(4) If $\beta_1, \beta_2 < \alpha \le \kappa$, α is a limit ordinal, $f_1 \in \mathscr{F}_{\beta_1\alpha}$, and $f_2 \in \mathscr{F}_{\beta_2\alpha}$ then there is some ordinal γ such that $\beta_1, \beta_2 < \gamma < \alpha$ and $\exists f_1' \in \mathscr{F}_{\beta_1\gamma}\, \exists f_2' \in \mathscr{F}_{\beta_2\gamma}\, \exists g \in \mathscr{F}_{\gamma\alpha}(f_1 = g \circ f_1'$ and $f_2 = g \circ f_2')$.

(5) $\bigcup\{f''\theta_\alpha : \alpha < \kappa, f \in \mathscr{F}_{\alpha\kappa}\} = \kappa^+$.

The morass is called *neat* if it satisfies the following stronger versions of (3) and (5).

(3′) For all $\alpha < \kappa$, $\mathscr{F}_{\alpha,\alpha+1}$ is a pair $\{\mathrm{id}_{\theta_\alpha}, f_\alpha\}$, where $\mathrm{id}_{\theta_\alpha}$ is the identity function with domain θ_α and for some ordinal $\sigma_\alpha < \theta_\alpha$, $f_\alpha \restriction \sigma_\alpha = \mathrm{id}_{\sigma_\alpha}$ and $f_\alpha(\sigma_\alpha) = \theta_\alpha$.

(5′) If $\beta < \alpha \le \kappa$ then $\theta_\alpha = \bigcup\{f''\theta_\beta : f \in \mathscr{F}_{\beta\alpha}\}$.

Every morass can be modified to make it neat.

Suppose $\alpha \le \kappa$, and that $f \in \mathscr{F}_{\gamma\alpha}$ and $g \in \mathscr{F}_{\beta\alpha}$ for some $\gamma < \beta < \alpha$. Let us say that g *dominates* f if there is some $h \in \mathscr{F}_{\gamma\beta}$ such that $f = g \circ h$. It is not hard to see that domination is a partial order, and clause (4) in the definition of simplified morass says that for α a limit ordinal it is directed. In some applications of morasses it is useful to pick out, for each limit ordinal $\alpha < \kappa$, a cofinal sequence of functions which is *linearly* ordered by domination. This is the motivation for the definition of simplified morasses with linear limits.

Suppose $\langle\langle\theta_\alpha : \alpha \le \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \le \kappa\rangle\rangle$ is a simplified $(\kappa, 1)$-morass, and for each limit ordinal $\alpha < \kappa$ we have a sequence $\vec{C}^\alpha = \langle\langle\beta_\delta^\alpha, f_\delta^\alpha\rangle : \delta < \tau^\alpha\rangle$ such that for all $\delta < \tau^\alpha$, $\beta_\delta^\alpha < \alpha$, and $f_\delta^\alpha \in \mathscr{F}_{\beta_\delta^\alpha\alpha}$.

Definition. The structure $\langle\langle\theta_\alpha : \alpha \le \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \le \kappa\rangle, \langle\vec{C}^\alpha : \alpha < \kappa, \alpha \text{ a limit ordinal}\rangle\rangle$ is called a *simplified* $(\kappa, 1)$-*morass with linear limits* if for every limit ordinal $\alpha < \kappa$ the following requirements are satisfied.

(1) (Linearity) If $\delta < \gamma < \tau^\alpha$ then $\beta_\delta^\alpha < \beta_\gamma^\alpha$ and $\exists g \in \mathscr{F}_{\beta_\delta^\alpha\beta_\gamma^\alpha}(f_\delta^\alpha = f_\gamma^\alpha \circ g)$.

(2) (Cofinality) If $\beta < \alpha$ and $f \in \mathscr{F}_{\beta\alpha}$ then there is some $\delta < \tau^\alpha$ such that $\beta < \beta^\alpha_\delta$ and $\exists g \in \mathscr{F}_{\beta\beta^\alpha_\delta}(f = f^\alpha_\delta \circ g)$.

(3) (Coherence) Suppose $\gamma < \tau^\alpha$ and γ is a limit ordinal. Let $\overline{\alpha} = \beta^\alpha_\gamma$. Then $\overline{\alpha}$ is a limit ordinal, $\tau^{\overline{\alpha}} = \gamma$, and $\forall\delta < \gamma\,(\beta^{\overline{\alpha}}_\delta = \beta^\alpha_\delta$ and $f^\alpha_\delta = f^\alpha_\gamma \circ f^{\overline{\alpha}}_\delta)$.

The sequence $\vec{C}^{\alpha}$ is called the *linearizing sequence* for α. We will say that a simplified morass $\langle\langle\theta_\alpha : \alpha \leq \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \leq \kappa\rangle\rangle$ *has linear limits* if there exist linearizing sequences $\vec{C}^{\alpha}$ for $\alpha < \kappa$, α a limit ordinal, satisfying (1)–(3) above.

To prove the proposition in the introduction we will also need our morass to have a complete amalgamation system. To simplify the definition of complete amalgamation system, let us assume that our morass is neat.

Suppose that for each $\alpha < \kappa$ we have an ordinal $\nu_\alpha < \kappa$, and sets $X_\alpha, Y_\alpha \subseteq \theta_\alpha$. We define sets A_α for $\alpha \leq \kappa$ by induction on α as follows:

$$A_0 = 0$$
$$A_{\alpha+1} = \{\langle\nu, f''X, f''Y\rangle : \langle\nu, X, Y\rangle \in A_\alpha \text{ and } f \in \mathscr{F}_{\alpha,\alpha+1}\} \cup \{\langle\nu_\alpha, X_\alpha, f''_\alpha Y_\alpha\rangle\}$$
$$A_\alpha = \{\langle\nu, f''X, f''Y\rangle : \exists\beta < \alpha(\langle\nu, X, Y\rangle \in A_\beta \text{ and } f \in \mathscr{F}_{\beta\alpha})\} \qquad \text{for } \alpha \text{ limit}$$

The sequence $\langle\langle\nu_\alpha, X_\alpha, Y_\alpha\rangle : \alpha < \kappa\rangle$ is called an *amalgamation system* if for all $\alpha < \kappa$ either $X_\alpha = Y_\alpha$, $\langle\nu_\alpha, X_\alpha, Y_\alpha\rangle \in A_\alpha$, or $\langle\nu_\alpha, Y_\alpha, X_\alpha\rangle \in A_\alpha$. The amalgamation system is *complete* if, in addition, whenever $\nu < \kappa$, $\mathscr{X} \subseteq P_\kappa(\kappa^+)$, and $|\mathscr{X}| = \kappa^+$ there are distinct $X, Y \in \mathscr{X}$ such that $\langle\nu, X, Y\rangle \in A_\kappa$. It is shown in [V3, Theorem 2.5] that if $\kappa = \lambda^+$ and $2^\lambda = \kappa$ then every simplified $(\kappa, 1)$-morass has a complete amalgamation system.

(1.3) **Parallels with** [SS1, §2]. For the most part the material in §2 of this paper is a straightforward adaptation to a recursive morass construction of the treatment of §2 of [SS1], except, of course that there is no need to formally develop historical conditions since for $\alpha < \omega_2$ the sequences $\langle\langle b_\beta, c_\beta\rangle : \beta \leq \alpha\rangle$ are rather special kinds of historical conditions. There are two important exceptions however. The first occurs in (2.3), in the proof of the lemma of (2.2). There, Velleman has simplified and improved the argument of [SS1, (2.15.4)], eliminating the need for the auxiliary partition there. This builds on a more thorough analysis of the histories of points, not limited to the finite set of stages when the history is nontrivial, as in [SS1] (though the latter notion continues to be of central importance here). It should be noted that this type of improvement could be carried out in the forcing context of [SS1]; it does not depend on the context of the morass construction, though it arises particularly naturally in this context.

The second difference is more substantial and *does* seem related to a difference between adjoining generic objects and constructing them from morasses. This is also related to the difference between the properties of the complete amalgamation system obtained from $2^{\aleph_1} = \aleph_2$ and the analogous properties that Miyamoto extracted from properties of the *generic* morass adjoined in his stage 1. The discussion of gray in (1.1) suggested that in amalgamations, a single gray edge may split into four: the original edge, its image and two "mixed" or diagonal edges. In [SS1] large homogeneous red sets are eliminated by coloring one of these mixed edges green. However, in successor stages of the recursive construction below, the complete amalgamation system typically gives us a *pair* of ascending gray edges $\langle x, y\rangle$, $\langle \overline{x}', \overline{y}'\rangle$ with $\gamma(x) = \gamma(\overline{x}')$, $\gamma(y) = \gamma(\overline{y}')$, and with the diagonal edges $\langle x, \overline{y}'\rangle$ and $\langle \overline{x}', y\rangle$ not red. Letting x', y' be the images of $\overline{x}'$, $\overline{y}'$ under the amalgamation map, we are to make the "mixed diagonal" edge $\langle x', y\rangle$ green *and* the "mixed diagonal" edge $\langle x, y'\rangle$ gray. The last point is the clue. Later on in the construction, the complete amalgamation system may mention the pair of edges $\langle x, y\rangle$, $\langle x', y'\rangle$, and the diagonal edges of this pair must not be red if we are to be able to continue the construction. Tracing through the definition of the complete amalgamation system, it is not too hard to see that a pair of edges, as above, mentioned by the complete amalgamation system, can ultimately be traced back to the split of a single edge in an amalgamation. However, in the recursive construction from a complete amalgamation system, unlike in the forcing context, it is not possible to work only with pairs resulting directly from the split of a single edge. Rather, it appears to be necessary to handle pairs resulting from a finite history of such splits. Miyamoto's analogue of the complete amalgamation system, abstracted from the properties given by genericity arguments, is sufficiently strong to let him handle only immediate splits. For this reason, we suspect that Miyamoto's property is strictly stronger than the existence of a complete amalgamation system, although we do not have a proof of this.

2. Proof of the proposition

(2.1) **Continuous paths.** To prove the proposition stated in the introduction we will need to define a new kind of morass structure similar to linear limits. Suppose κ is a regular uncountable cardinal, and $\langle\langle\theta_\alpha : \alpha \le \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \le \kappa\rangle\rangle$ is a simplified $(\kappa, 1)$-morass. Motivated by [M], we make the following definition.

Definition. A sequence $\langle g_\beta : \beta < \alpha\rangle$ is a *continuous path at* α if

(1) $\forall\beta < \alpha(g_\beta \in \mathscr{F}_{\beta\alpha})$.
(2) (Linearity) If $\beta < \gamma < \alpha$ then $\exists f \in \mathscr{F}_{\beta\gamma}(g_\beta = g_\gamma \circ f)$.
(3) (Cofinality) If α is a limit ordinal, $\beta < \alpha$, and $f \in \mathscr{F}_{\beta\alpha}$, then there is some δ such that $\beta < \delta < \alpha$ and $\exists f' \in \mathscr{F}_{\beta\delta}(f = g_\delta \circ f')$.
(4) (Continuity) If $\beta < \gamma < \alpha$, γ is a limit ordinal, and $f \in \mathscr{F}_{\beta\gamma}$, then there is some δ such that $\beta < \delta < \gamma$ and $\exists f' \in \mathscr{F}_{\beta\delta}(g_\gamma \circ f = g_\delta \circ f')$.

The simplified morass has *continuous paths* if it has a continuous path at α for every $\alpha < \kappa$.

In the proof of the proposition we will need our morass to have continuous paths. The next lemma guarantees that such paths exist.

Lemma. *Suppose* $\langle\langle\theta_\alpha : \alpha \leq \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \leq \kappa\rangle\rangle$ *is a simplified* $(\kappa, 1)$*-morass which has linear limits,* $\beta < \alpha < \kappa$, *and* $f \in \mathscr{F}_{\beta\alpha}$. *Then there is a continuous path* $\langle g_\gamma : \gamma < \alpha\rangle$ *at* α *such that* $g_\beta = f$. *Thus, every simplified morass which has linear limits also has continuous paths.*

Proof. The proof is by induction on α.

Case 1. α is a successor ordinal, say $\alpha = \alpha' + 1$. If $\beta = \alpha'$, then by inductive hypothesis let $\langle g'_\gamma : \gamma < \alpha'\rangle$ be a continuous path at α'. Now for $\gamma < \alpha'$ let $g_\gamma = f \circ g'_\gamma$, and let $g_\beta = f$. It is easy to verify that $\langle g_\gamma : \gamma < \alpha\rangle$ is a continuous path at α, as required.

Now suppose $\beta < \alpha'$. Choose $f_1 \in \mathscr{F}_{\beta\alpha'}$ and $f_2 \in \mathscr{F}_{\alpha'\alpha}$ such that $f = f_2 \circ f_1$, and by the inductive hypothesis let $\langle g'_\gamma : \gamma < \alpha'\rangle$ be a continuous path at α' such that $g'_\beta = f_1$. For $\gamma < \alpha'$ let $g_\gamma = f_2 \circ g'_\gamma$, and let $g_{\alpha'} = f_2$. Then $\langle g_\gamma : \gamma < \alpha\rangle$ is a continuous path at α, and $g_\beta = f_2 \circ g'_\beta = f_2 \circ f_1 = f$.

Case 2. α is a limit ordinal. Let $\langle\langle\beta_\delta, f_\delta\rangle : \delta < \tau\rangle$ be the linearizing sequence for α. Then there is some $\zeta < \tau$ and some $f' \in \mathscr{F}_{\beta\beta_\zeta}$ such that $f = f_\zeta \circ f'$. By inductive hypothesis choose a continuous path $\langle g'_\gamma : \gamma < \beta_\zeta\rangle$ at β_ζ such that $g'_\beta = f'$, and for each $\delta \in \tau \setminus \zeta$ choose a continuous path $\langle g^\delta_\gamma : \gamma < \beta_{\delta+1}\rangle$ at $\beta_{\delta+1}$ such that $g^\delta_{\beta_\delta} = (f_{\delta+1})^{-1} \circ f_\delta$. Now we combine all of these continuous paths to get a continuous path at α as follows. For $\gamma < \beta_\zeta$ let $g_\gamma = f_\zeta \circ g'_\gamma$. If $\beta_\delta \leq \gamma < \beta_{\delta+1}$ for some $\delta \in \tau \setminus \zeta$ then let $g_\gamma = f_{\delta+1} \circ g^\delta_\gamma$. It is routine to verify now that $\langle g_\gamma : \gamma < \alpha\rangle$ is a continuous path at α, and $g_\beta = f_\zeta \circ g'_\beta = f_\zeta \circ f' = f$. □

Note that continuous paths are quite similar to linearizing sequences, but differ in two respects. First of all, a continuous path at α must include mappings to level α from *all* levels below α, while a linearizing sequence only picks out mappings from certain levels below α. Secondly, linearizing sequences must satisfy the coherence property, which, when combined with the cofinality property, can easily be shown to imply the continuity property. Continuous paths need only satisfy continuity. In fact, it is *impossible* for continuous paths to cohere:

Proposition. *Suppose* $\langle\langle\theta_\alpha : \alpha \leq \kappa\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \leq \kappa\rangle\rangle$ *is a simplified* $(\kappa, 1)$*-morass, and for each* $\alpha < \kappa$, $\langle g^\alpha_\beta : \beta < \alpha\rangle$ *is a continuous path at* α. *Then it is not the case that whenever* $\beta < \gamma < \alpha < \kappa$ *and* γ *and* α *are both limit ordinals,* $g^\alpha_\beta = g^\alpha_\gamma \circ g^\gamma_\beta$. *Thus, continuous paths cannot be linearizing sequences.*

Proof. Suppose $\beta < \kappa$ and $\zeta < \theta_\beta$. It is not hard to show that there is at most one ordinal $\tau < \kappa^+$ such that for every limit ordinal $\gamma < \kappa$, if $\beta < \gamma$ then $\exists f \in \mathscr{F}_{\gamma\kappa}(f(g^\gamma_\beta(\zeta)) = \tau)$. Let $\tau_{\beta\zeta}$ be the unique such ordinal τ, if there is one. Now choose $\tau < \kappa^+$ such that for all $\beta < \kappa$ and $\zeta < \theta_\beta$, $\tau \neq \tau_{\beta\zeta}$. For each $\beta < \kappa$ and $\zeta < \theta_\beta$ let $s(\beta, \zeta)$ be the least ordinal witnessing the fact that $\tau \neq \tau_{\beta\zeta}$. In other words, $s(\beta, \zeta)$ is the least limit ordinal $\gamma < \kappa$ such that $\beta < \gamma$ and $\forall f \in \mathscr{F}_{\gamma\kappa}(f(g^\gamma_\beta(\zeta)) \neq \tau)$.

Choose $\delta < \kappa$, $\overline{\tau} < \theta_\delta$, and $h \in \mathscr{F}_{\delta\kappa}$ such that $h(\overline{\tau}) = \tau$. Let α be a limit ordinal such that $\delta < \alpha < \kappa$ and $\forall\beta < \alpha\forall\zeta < \theta_\beta(s(\beta, \zeta) < \alpha)$, and choose $h_1 \in \mathscr{F}_{\delta\alpha}$ and $h_2 \in \mathscr{F}_{\alpha\kappa}$ such that $h = h_2 \circ h_1$. By the definition of continuous path, there is some β such that $\delta < \beta < \alpha$ and some $h' \in \mathscr{F}_{\delta\beta}$ such that $h_1 = g^\alpha_\beta \circ h'$. Let $\zeta = h'(\overline{\tau})$, $\gamma = s(\beta, \zeta) < \alpha$, and $f = h_2 \circ g^\alpha_\gamma \in \mathscr{F}_{\gamma\kappa}$. By the definition of $s(\beta, \zeta)$, $f(g^\gamma_\beta(\zeta)) \neq \tau$. But $h_2(g^\alpha_\beta(\zeta)) = h_2(g^\alpha_\beta(h'(\overline{\tau}))) = h_2(h_1(\overline{\tau})) = h(\overline{\tau}) = \tau$, so $h_2(g^\alpha_\beta(\zeta)) \neq f(g^\gamma_\beta(\zeta)) = h_2(g^\alpha_\gamma(g^\gamma_\beta(\zeta)))$. Therefore $g^\alpha_\beta(\zeta) \neq g^\alpha_\gamma(g^\gamma_\beta(\zeta))$, so $g^\alpha_\beta \neq g^\alpha_\gamma \circ g^\gamma_\beta$. □

Donder's proof in [D] that if κ is weakly compact then a simplified $(\kappa, 1)$-morass cannot have linear limits makes essential use of the coherence of the linearizing sequences. In light of the observations above it is therefore natural to ask if a simplified $(\kappa, 1)$-morass can have continuous paths if κ is weakly compact. In particular, for which regular cardinals κ do the natural simplified $(\kappa, 1)$-morasses in L have continuous paths? We do not know the answers to these questions.

(2.2) **Proof of the proposition.** Let $\langle\langle\theta_\alpha : \alpha \leq \omega_2\rangle, \langle\mathscr{F}_{\alpha\beta} : \alpha < \beta \leq \omega_2\rangle\rangle$ be a simplified $(\omega_2, 1)$-morass which has linear limits. By [V2, Theorem 4.2], we may assume that the morass is neat. By the lemma of (2.1) we know that the morass has continuous paths, and since we are assuming that $2^{\aleph_1} = \aleph_2$, we also know that it has a complete amalgamation system. Let $\langle\langle\nu_\alpha, X_\alpha, Y_\alpha\rangle : \alpha < \omega_2\rangle$ be a complete amalgamation system, and let A_α be defined as in (1.2) above.

To prove that $\omega_3\omega_1 \nrightarrow (\omega_3\omega_1, 3)^2$ we must produce a function $c : (\omega_1 \times \omega_3)^2 \to \{\text{red}, \text{green}\}$ with no large homogeneous red set and no green triangle. As usual, we construct this function from small pieces which are assigned to the levels of the morass. The definition of these small pieces is motivated by the definition of P_{basic} in [SS1, §2].

For each $\alpha \leq \omega_2$ we will choose $c_\alpha : (\omega_1 \times \theta_\alpha)^2 \to \{\text{gray}, \text{red}, \text{green}\}$ and $b_\alpha : (\omega_1 \times \theta_\alpha)^2 \to \omega_1$ such that

(1) If $c_\alpha(x, y) = \text{green}$ then $\langle x, y\rangle$ is a descending edge.
(2) $c_\alpha(x, y) = \text{red}$ iff $\exists z(c_\alpha(x, z) = c_\alpha(y, z) = \text{green})$.
(3) If $c_\alpha(x, z) = c_\alpha(y, z) = \text{green}$ then $\gamma(z) < b_\alpha(x, y)$.

There are three more conditions on b_α and c_α, but we will need some notation before we can state them. Suppose $\theta < \omega_2$ and $f : \theta \to \omega_3$ is an order preserving function. Define $\tilde{f} : \omega_1 \times \theta \to \omega_1 \times \omega_3$ by $\tilde{f}(\gamma, \rho) = \langle \gamma, f(\rho) \rangle$. We will also require

(4) If $\beta < \alpha$, $f \in \mathscr{F}_{\beta\alpha}$, and $\langle x, y \rangle \in (\omega_1 \times \theta_\beta)^2$ then $c_\alpha(\tilde{f}(x), \tilde{f}(y)) = c_\beta(x, y)$ and $b_\alpha(\tilde{f}(x), \tilde{f}(y)) = b_\beta(x, y)$.

(5) There is at most one pair $x, y \in \omega_1 \times \theta_\alpha$ such that $\sigma_\alpha < \rho(x), \rho(y)$ and $c_{\alpha+1}(\tilde{f}_\alpha(x), y) = \text{green}$. If there is such a pair, we will call $\langle \tilde{f}_\alpha(x), y \rangle$ the *new green edge at stage* $\alpha + 1$.

Finally, we need a condition specifying how we will use the complete amalgamation system to prevent large homogeneous red sets from being formed. Fix a bijection $j : \omega_1 \to \{\langle \gamma, \delta \rangle \in \omega_1 \times \omega_1 : \gamma < \delta\}$. Suppose $\nu < \omega_1$, $j(\nu) = \langle \gamma, \delta \rangle$, and $X = \{\alpha, \beta\}$ where $\alpha < \beta < \omega_3$. We will say that $\langle \nu, X \rangle$ is a *name* for the ascending edge $\langle \langle \gamma, \alpha \rangle, \langle \delta, \beta \rangle \rangle \in (\omega_1 \times \omega_3)^2$. Our last requirement is:

(6) Suppose $\langle \nu, X, Y \rangle \in A_\alpha$, $\langle \nu, X \rangle$ is a name for an edge $\langle x, y \rangle$, $\langle \nu, Y \rangle$ is a name for another edge $\langle x', y' \rangle$, and $c_\alpha(x, y) = c_\alpha(x', y') = \text{gray}$. Then $c_\alpha(x, y') \neq \text{red}$ and $c_\alpha(x', y) \neq \text{red}$. Furthermore, if $\rho(y) < \rho(x')$ then $c_\alpha(x', y) = \text{green}$ and if $\rho(y') < \rho(x)$ then $c_\alpha(x, y') = \text{green}$.

We choose c_α and b_α by induction on α. We can let b_0 be arbitrary, and for all $\langle x, y \rangle \in (\omega_1 \times \theta_0)^2$ we let $c_0(x, y) = \text{gray}$. At limit stages condition (4) above completely determines c_α and b_α, and it is easy to verify that (1)–(6) are preserved.

Now suppose c_α and b_α have been chosen and we wish to choose $c_{\alpha+1}$ and $b_{\alpha+1}$. It turns out that even in this case the inductive conditions (1)–(6) leave us very little choice about how to proceed. Since $\mathscr{F}_{\alpha,\alpha+1}$ is an amalgamation pair $\{\mathrm{id}_{\theta_\alpha}, f_\alpha\}$, condition (4) determines the values of $c_{\alpha+1}$ and $b_{\alpha+1}$ on all edges in $(\omega_1 \times \theta_{\alpha+1})^2$ except those of the form $\langle x, \tilde{f}_\alpha(y) \rangle$ or $\langle \tilde{f}_\alpha(x), y \rangle$, where $\sigma_\alpha < \rho(x), \rho(y)$. We will call these edges "mixed edges." According to condition (5), at most one descending mixed edge can be green, and condition (2) then determines the colors of all remaining mixed edges. Once the colors of all edges have been determined, it is easy to define $b_{\alpha+1}$ on mixed edges so that (3) will hold. Thus the only difficult decision we have to make is which descending mixed edge, if any, should be colored green.

We let the reader check that if no mixed edges are colored green then (1)–(5) will hold. As in [SS1], we will call this the *no-green amalgamation*. Now suppose $\langle x, y \rangle \in (\omega_1 \times \theta_\alpha)^2$, $\sigma_\alpha < \rho(x), \rho(y)$, and $c_\alpha(x, y) \neq \text{red}$. We will call such an edge $\langle x, y \rangle$ *acceptable*. It is not hard to check that if $\langle x, y \rangle$ is acceptable then we can let $\langle \tilde{f}_\alpha(x), y \rangle$ be the new green edge at stage $\alpha + 1$ and (1)–(5) will be preserved. The acceptability of $\langle x, y \rangle$ is needed to guarantee that (2) will hold for the edge $\langle \tilde{f}_\alpha(x), y \rangle$. In the terminology of [SS1], this would be called the $\{\langle x, y \rangle\}$ *amalgamation*. Thus, as long as we use either the no-green amalgamation or the $\{\langle x, y \rangle\}$ amalgamation for some acceptable

$\langle x, y\rangle$, all we have to worry about is making sure that (6) holds at stage $\alpha+1$.

Clearly the only element of $A_{\alpha+1}$ that could cause a problem in (6) is the triple $\langle \nu_\alpha, X_\alpha, f''_\alpha Y_\alpha\rangle$. If the hypothesis of (6) is not satisfied for this triple then we can simply use the no-green amalgamation to define $c_{\alpha+1}$ and $b_{\alpha+1}$, and (6) will hold. Now suppose the hypothesis of (6) is satisfied for the triple $\langle \nu_\alpha, X_\alpha, f''_\alpha Y_\alpha\rangle$. Then $\langle \nu_\alpha, X_\alpha\rangle$ is a name for an edge $\langle x, y\rangle$ and $\langle \nu_\alpha, f''_\alpha Y_\alpha\rangle$ is a name for another edge $\langle x', y'\rangle$. Clearly $\langle \nu_\alpha, Y_\alpha\rangle$ must be a name for an edge $\langle \overline{x}', \overline{y}'\rangle$, where $\widetilde{f}_\alpha(\overline{x}') = x'$ and $\widetilde{f}_\alpha(\overline{y}') = y'$. Also, by (4) we must have $c_{\alpha+1}(x, y) = c_\alpha(x, y)$ and $c_{\alpha+1}(x', y') = c_\alpha(\overline{x}', \overline{y}')$, so the hypothesis of (6) can only be satisfied if $c_\alpha(x, y) = c_\alpha(\overline{x}', \overline{y}') = \text{gray}$.

By the definition of complete amalgamation system, either $\langle \nu_\alpha, X_\alpha, Y_\alpha\rangle \in A_\alpha$, $\langle \nu_\alpha, Y_\alpha, X_\alpha\rangle \in A_\alpha$, or $X_\alpha = Y_\alpha$. By (6) for stage α, in all three cases we have $c_\alpha(x, \overline{y}') \neq \text{red}$ and $c_\alpha(\overline{x}', y) \neq \text{red}$, so by (2) there is no $z \in \omega_1 \times \theta_\alpha$ such that either $c_\alpha(x, z) = c_\alpha(\overline{y}', z) = \text{green}$ or $c_\alpha(\overline{x}', z) = c_\alpha(y, z) = \text{green}$. Thus if we use the no-green amalgamation to define $c_{\alpha+1}$ then there will be no z such that either $c_{\alpha+1}(x, z) = c_{\alpha+1}(y', z) = \text{green}$ or $c_{\alpha+1}(x', z) = c_{\alpha+1}(y, z) = \text{green}$, so we will have $c_{\alpha+1}(x, y') \neq \text{red}$ and $c_{\alpha+1}(x', y) \neq \text{red}$.

The only case in which the no-green amalgamation cannot be used is if $\sigma_\alpha < \rho(\overline{x}'), \rho(y)$. In this case $\langle x', y\rangle$ is a mixed descending edge, and by the last sentence in (6) we must color it green. But in this case $\langle \overline{x}', y\rangle$ is acceptable, so we can use the $\{\langle \overline{x}', y\rangle\}$ amalgamation, in which $\langle x', y\rangle$ is the new green edge. As above, $c_{\alpha+1}(x, y') \neq \text{red}$, so (6) is satisfied.

This completes the inductive construction of the c_α's and b_α's. We now define $c : (\omega_1 \times \omega_3)^2 \to \{\text{red, green}\}$ by $c(x, y) = \text{green}$ iff $c_{\omega_2}(x, y) = \text{green}$. It is clear by (2) for ω_2 that this coloring has no green triangles. To show that there are no large homogeneous red sets we will need the following lemma.

Lemma. *There is no set H which is homogeneous red for c_{ω_2} such that $|\{\gamma(x) : x \in H\}| = \aleph_1$.*

We will prove the lemma in (2.3) below. First we will see how to use it to finish the proof of the proposition. Suppose H is a large homogeneous red set for c. Choose sets $H_\alpha \subseteq H$ for $\alpha < \omega_3$ such that

(1) $|H_\alpha| = \aleph_1$.
(2) If $x, y \in H_\alpha$ and $x \neq y$ then $\gamma(x) \neq \gamma(y)$. Also, if $\gamma(x) < \gamma(y)$ then $\rho(x) < \rho(y)$.
(3) If $\alpha < \beta < \omega_3$, $x \in H_\alpha$, and $y \in H_\beta$, then $\rho(x) < \rho(y)$.

By the Lemma, no H_α can be homogeneous red for c_{ω_2}, so each must include an edge which is colored gray. Thus we can find ordinals $\gamma < \delta < \omega_1$, a set $S \subseteq \omega_3$ such that $|S| = \aleph_3$, and $x_\alpha, y_\alpha \in H_\alpha$ for each $\alpha \in S$ such that $\gamma(x_\alpha) = \gamma$, $\gamma(y_\alpha) = \delta$, and $c_{\omega_2}(x_\alpha, y_\alpha) = \text{gray}$. Choose ν such that $j(\nu) = \langle \gamma, \delta\rangle$, and let $\mathscr{X} = \{\{\rho(x_\alpha), \rho(y_\alpha)\} : \alpha \in S\}$. By the definition of complete amalgamation system, there must be distinct $X, Y \in \mathscr{X}$ such that

$\langle \nu, X, Y\rangle \in A_{\omega_2}$. But $\langle \nu, X\rangle$ and $\langle \nu, Y\rangle$ are names for edges $\langle x_\alpha, y_\alpha\rangle$ and $\langle x_\beta, y_\beta\rangle$ for some distinct $\alpha, \beta \in S$. Thus by inductive hypothesis (6) in the construction of the c_α's, either $c_{\omega_2}(x_\alpha, y_\beta) = \text{green}$ or $c_{\omega_2}(x_\beta, y_\alpha) = \text{green}$, contradicting the assumption that H was a large homogeneous red set. $\square$

(2.3) **Proof of the lemma of (2.2).** Suppose H is homogeneous red for c_{ω_2} and $|\{\gamma(x) : x \in H\}| = \aleph_1$. By thinning H if necessary we may assume that if $x, y \in H$ and $x \neq y$ then $\gamma(x) \neq \gamma(y)$ and $|H| = \aleph_1$. By [V3, Lemma 2.3] we can find a limit ordinal $\alpha < \omega_2$ and a set $\overline{H} \subseteq \omega_1 \times \theta_\alpha$ such that for some $f \in \mathscr{F}_{\alpha\omega_2}$, $\tilde{f}''\overline{H} = H$. Clearly $\overline{H}$ is homogeneous red for c_α, $|\overline{H}| = \aleph_1$, and if $x, y \in \overline{H}$ and $x \neq y$ then $\gamma(x) \neq \gamma(y)$. From now on we work entirely with $\overline{H}$.

Let $\langle g_\beta : \beta < \alpha\rangle$ be a continuous path at α, and suppose $x \in \overline{H}$. By the neatness of the simplified morass and [V1, Lemma 3.2], for each $\beta < \alpha$ there is a unique $x_\beta \in \omega_1 \times \theta_\beta$ such that for some $f \in \mathscr{F}_{\beta\alpha}$, $\tilde{f}(x_\beta) = x$.

Claim 1. Suppose $\delta < \beta < \alpha$ and $\widetilde{g_\delta}(z) = \widetilde{g_\beta}(x_\beta)$ for some $z \in \omega_1 \times \theta_\delta$. Then $z = x_\delta$.

Proof. Choose $f \in \mathscr{F}_{\delta\beta}$ such that $g_\delta = g_\beta \circ f$. Then $\widetilde{g_\beta}(x_\beta) = \widetilde{g_\delta}(z) = \widetilde{g_\beta}(\tilde{f}(z))$, so since $\widetilde{g_\beta}$ is one-to-one, $\tilde{f}(z) = x_\beta$. Now choose $h \in \mathscr{F}_{\beta\alpha}$ such that $\tilde{h}(x_\beta) = x$. Then $\widetilde{h \circ f}(z) = x$, so $z = x_\delta$.

Claim 2. If $\beta < \alpha$ and β is a limit ordinal then for all sufficiently large $\delta < \beta$, $\widetilde{g_\delta}(x_\delta) = \widetilde{g_\beta}(x_\beta)$. Also, for all sufficiently large $\delta < \alpha$, $\widetilde{g_\delta}(x_\delta) = x$.

Proof. By the neatness of the simplified morass, we can choose $\eta < \beta$, $z \in \omega_1 \times \theta_\eta$, and $f \in \mathscr{F}_{\eta\beta}$ such that $\tilde{f}(z) = x_\beta$. By (2) and (4) in the definition of continuous path, for all sufficiently large $\delta < \beta$ there is some $f' \in \mathscr{F}_{\eta\delta}$ such that $g_\beta \circ f = g_\delta \circ f'$, so $\widetilde{g_\beta}(x_\beta) = \widetilde{g_\beta}(\tilde{f}(z)) = \widetilde{g_\delta}(\widetilde{f'}(z))$. Now by Claim 1, $\widetilde{f'}(z) = x_\delta$, so $\widetilde{g_\beta}(x_\beta) = \widetilde{g_\delta}(x_\delta)$ as required. The proof of the second statement in the claim is very similar, using (3) in the definition of continuous path instead of (4).

Let $\beta_0^x = 0$ and $a_0^x = \widetilde{g_0}(x_0)$. For each $n \in \omega$ if β_n^x and a_n^x are defined then let β_{n+1}^x be the least ordinal $\beta > \beta_n^x$ such that $\widetilde{g_\beta}(x_\beta) \neq a_n^x$, if there is one, and let $a_{n+1}^x = \widetilde{g_\beta}(x_\beta)$ for this β. Note that by Claim 2, if β_{n+1}^x is defined then it must be a successor ordinal. If β_n^x and a_n^x were defined for every $n \in \omega$ then, applying Claim 2 to $\beta = \bigcup_{n\in\omega} \beta_n^x$, for all sufficiently large n we would have $a_n^x = \widetilde{g_\beta}(x_\beta)$, contradicting the definition of the β_n^x's and a_n^x's. Thus there is a largest n for which β_n^x and a_n^x are defined. Clearly if $\beta_i^x \le \delta < \beta_{i+1}^x$ then $\widetilde{g_\delta}(x_\delta) = a_i^x$, and similarly if $\beta_n^x \le \delta$ then $\widetilde{g_\delta}(x_\delta) = a_n^x$. Also, by the second part of Claim 2, $a_n^x = x$. Thus, $\{a_0^x, a_1^x, \dots, a_n^x = x\}$ is the set of all possible values for $\widetilde{g_\delta}(x_\delta)$ for $\delta < \alpha$. For the reader who is familiar with [SS1] we note that $\{\beta_0^x, \beta_1^x, \dots, \beta_n^x\}$ corresponds to $\mathrm{hist}_1(x)$ in [SS1] and $\{a_0^x, a_1^x, \dots, a_n^x\}$ corresponds to $\mathrm{hist}_2(x)$.

By thinning $\overline{H}$ if necessary we may assume that there is a fixed n such that for all $x \in \overline{H}$, β_n^x and a_n^x are defined but β_{n+1}^x and a_{n+1}^x are not. Thinning $\overline{H}$ further, we may also assume:

(1) Suppose $x \in \overline{H}$, $0 < i \le n$, and $\langle v, w\rangle$ is the new green edge at stage β_i^x. Then for every $y \in \overline{H}$, if $\gamma(y) > \gamma(x)$ then $\gamma(y) > \gamma(w)$.

(2) Suppose $x, y, z \in \overline{H}$, $\gamma(x) < \gamma(y) < \gamma(z)$, and $i, j \le n$. Then $b_\alpha(a_i^x, a_j^y) < \gamma(z)$.

Choose $z \in \overline{H}$ such that $\gamma(z)$ is the $(n+2)$th element of $\{\gamma(x) : x \in \overline{H}\}$. Now suppose $x \in \overline{H}$ and $\gamma(x) < \gamma(z)$. Since $\overline{H}$ is homogeneous red for c_α, $c_\alpha(x, z) = \text{red}$, and therefore, for sufficiently large $\beta < \alpha$, $c_\beta(x_\beta, z_\beta) = \text{red}$. Let β be the least ordinal such that $c_\beta(x_\beta, z_\beta) = \text{red}$. We will call β the *stage at which* $\langle x, z\rangle$ *first became red.* Since $\mathrm{range}(c_0) = \{\text{gray}\}$, $\beta > 0$, so β must be a successor ordinal, say $\beta = \bar\beta + 1$.

By inductive hypothesis (2) we can find some $w \in \omega_1 \times \theta_\beta$ such that $c_\beta(x_\beta, w) = c_\beta(z_\beta, w) = \text{green}$. In fact, one of these edges must be the new green edge at stage β, since $c_{\bar\beta}(x_{\bar\beta}, z_{\bar\beta}) \ne \text{red}$. Also, by inductive hypothesis (4) $c_\beta(x_\beta, z_\beta) = c_\alpha(\widetilde{g_\beta}(x_\beta), \widetilde{g_\beta}(z_\beta)) = \text{red}$ and $c_{\bar\beta}(x_{\bar\beta}, z_{\bar\beta}) = c_\alpha(\widetilde{g_{\bar\beta}}(x_{\bar\beta}), \widetilde{g_{\bar\beta}}(z_{\bar\beta})) \ne \text{red}$, so either $\widetilde{g_\beta}(x_\beta) \ne \widetilde{g_{\bar\beta}}(x_{\bar\beta})$ or $\widetilde{g_\beta}(z_\beta) \ne \widetilde{g_{\bar\beta}}(z_{\bar\beta})$. Thus either $\beta = \beta_i^x$ or $\beta = \beta_i^z$ for some $i \le n$, $i > 0$. But if $\beta = \beta_i^x$ then by (1) above we must have $\gamma(z_\beta) = \gamma(z) > \gamma(w)$, which is a contradiction. Therefore $\beta = \beta_i^z$. Thus we have shown that, keeping z fixed and letting x vary, there are only n possibilities for the stage at which $\langle x, z\rangle$ first became red.

Since $\gamma(z)$ is the $(n+2)$th element of $\{\gamma(x) : x \in \overline{H}\}$, we can find some $x, y \in \overline{H}$ such that $\gamma(x) < \gamma(y) < \gamma(z)$ and the edges $\langle x, z\rangle$ and $\langle y, z\rangle$ both first became red at the same stage β. Since there can be only one new green edge at stage β, there must be some $w \in \omega_1 \times \theta_\beta$ such that $\langle z_\beta, w\rangle$ is the new green edge at stage β, and also $c_\beta(x_\beta, w) = c_\beta(y_\beta, w) = \text{green}$. By inductive hypothesis (3) it follows that $b_\beta(x_\beta, y_\beta) > \gamma(w) > \gamma(z_\beta) = \gamma(z)$. But $b_\beta(x_\beta, y_\beta) = b_\alpha(\widetilde{g_\beta}(x_\beta), \widetilde{g_\beta}(y_\beta)) = b_\alpha(a_i^x, a_j^y)$ for some $i, j \le n$, so this contradicts (2) above. This contradiction completes the proof of the lemma. □

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Added in proof. We are grateful to H. D. Donder for pointing out the corrected formulation of the theorem of the introduction. We originally had the stronger (so much so, that it is consistently false) formulation where the conclusion simply asserts that $\aleph_3$ is $(\text{inaccessible})^L$. This was based on our erroneous idea that if $\aleph_3$ is a $(\text{successor cardinal})^L$, then there is a simplified $(\omega_2, 1)$-morass with linear limits, the idea being that one can be built in $L[A]$, where $A \subseteq \omega_2$ and $\aleph_3 = (\aleph_3)^{L[A]}$. Indeed, for such A, one *can* build a simplified $(\omega_2, 1)$-morass in $L[A]$, but not necessarily one with linear limits, as the following example of Donder shows.

Suppose $V = L$, and that κ is weakly compact. Levy-collapse κ to be $\aleph_2$, coding the generic by $A \subseteq \kappa$. Donder observed that in $L[A]$, Todorcevic's principle $\square(\aleph_2)$ fails. $\square(\aleph_2)$ is like the usual $\square_{\aleph_1}$, except that the C_α's are not required to have order type $\leq \omega_1$. The proof is like the classical proof that if a Mahlo cardinal is Levy-collapsed to λ^+ then $\square_\lambda$ fails, except that an argument using the Π^1_1-indescribability of κ in the ground model is required to show that for many $\alpha < \kappa$ which are regular in V and will have cofinality $\aleph_1$ in the extension, C_α cannot lie in the intermediate model where α has become $\aleph_2$. On the other hand, by the coherence property, C_α is uniquely determined when, in the extension, α has cofinality $\aleph_1$, and so, by the homogeneity of the rest of the Levy collapse, would have to lie in the intermediate model if it were in the extension. The counterexample is completed by Donder's proof in [D], building on work of the second author, that if there is a simplified $(\theta, 1)$-morass with linear limits then $\square(\theta)$ holds.

Thus, some restrictions are necessary on A in order to be able to build the desired simplified $(\omega_2, 1)$-morass with linear limits in $L[A]$. The weakest restriction which is known to be sufficient (and, in view of the preceding, quite likely the weakest sufficient restriction) was communicated to us by Donder: there exists $B \subseteq \omega_2$, $B \in L[A]$, and Π^1_1 formula $\Phi(B)$ such that $(L_{\omega_2}[A] \models \Phi(B))^{L[A]}$, but for all $\alpha < \omega_2$, $(L_\alpha[A \cap \alpha] \models \neg\Phi(B \cap \alpha))^{L[A \cap \alpha]}$. When this occurs, Donder's construction in [D] *can* be carried out. Finally, if $\aleph_2$ and $\aleph_3$ are both $(\text{successor cardinals})^L$, then there is $A \subseteq \omega_2$, such that $\aleph_3 = (\aleph_3)^{L[A]}$ and $\aleph_2 = (\aleph_2)^{L[A \cap \omega_1]}$. But such an A can easily be seen to have Donder's property.

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