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A CHARACTERIZATION OF SUBMETACOMPACTNESS IN TERMS OF PRODUCTS

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ABSTRACT. A space X is said to be *suborthocompact* if for every open cover $\mathscr{U}$ of X there is a sequence $\{\mathscr{V}_n\}$ of open refinements of $\mathscr{U}$ such that for each $x \in X$ there is some $n \in \omega$ such that $\bigcap\{V \in \mathscr{V}_n : x \in V\}$ is a neighborhood of x in X. It is proved that a Tychonoff space X is submetacompact if and only if the product $X \times \beta X$ is suborthocompact.

0. INTRODUCTION

According to Tamano's Theorem [T1], a Tychonoff space X is paracompact if and only if the product $X \times \beta X$ is normal, where βX denotes the Stone-Čech compactification of X. As a nice analogue, Junnila [Ju1] proved that a Tychonoff space X is metacompact if and only if the product $X \times \beta X$ is orthocompact. Our purpose in this paper is to give another analogue for submetacompactness.

Main Theorem. *A Tychonoff space X is submetacompact if and only if the product $X \times \beta X$ is suborthocompact.*

In §1, we will precisely introduce the concept of suborthocompact spaces, whose class contains the classes of submetacompact spaces and orthocompact spaces. In §2, our Main Theorem will be proved in a slightly generalized form. Second, we consider Katuta-Junnila's problem, which was first raised in [K, Problems 2.6 and 2.7]. Jiang [J] gave a partial answer to this problem. In §3, we will give an extension of his result, using suborthocompactness.

All spaces dealt with here are topological spaces. No separation axioms are assumed, unless otherwise stated.

1. SUBMETACOMPACTNESS AND SUBORTHOCOMPACTNESS

Let X be a space. For two covers $\mathscr{U}$ and $\mathscr{V}$ of X, $\mathscr{V}$ is a *refinement* of $\mathscr{U}$ (or $\mathscr{V}$ *refines* $\mathscr{U}$) if each member of $\mathscr{V}$ is contained in some member of

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$\mathscr{U}$. For a subset Y of X, $\mathrm{Cl}\, Y$ or $\mathrm{Cl}_X Y$ denotes the closure of Y in X.

Let $\mathscr{V}$ be a collection of subsets of a space X. For each $x \in X$, let

$$\mathscr{V}(x) = \{V \in \mathscr{V} : x \in V\},$$
$$\bigcap \mathscr{V}(x) = \bigcap \{V \in \mathscr{V} : x \in V\},$$

and

$$\mathrm{St}(x, \mathscr{V}) = \bigcup \{V \in \mathscr{V} : x \in V\} = \bigcup \mathscr{V}(x).$$

A space X is *submetacompact* (or *θ-refinable*) if for every open cover $\mathscr{U}$ of X there is a sequence $\{\mathscr{V}_n\}$ of open refinements of $\mathscr{U}$ such that, for each $x \in X$, there is some $n \in \omega$ such that $\mathscr{V}_n(x)$ is finite (i.e., $\mathscr{V}_n$ is point-finite at x).

However, we will prefer to make use of the following characterization of submetacompactness.

Theorem 1.1 (Junnila [Ju2]). *A space X is submetacompact if and only if, for every open cover $\mathscr{U}$ of X, there is a sequence $\{\mathscr{W}_n\}$ of open covers of X such that, for each $x \in X$, there are an $n \in \omega$ and a finite subcollection $\mathscr{U}_x$ of $\mathscr{U}$ such that* $\mathrm{St}(x, \mathscr{W}_n) \subset \bigcup \mathscr{U}_x$ *and* $x \in \bigcap \mathscr{U}_x$.

A space X is *orthocompact* if every open cover $\mathscr{U}$ of X has an open refinement $\mathscr{V}$ such that $\bigcap \mathscr{V}(x)$ is open in X for each $x \in X$ (i.e., $\mathscr{V}$ is interior-preserving).

Now, we introduce the following concept:

Definition 1.2. A space X is said to be *suborthocompact* if, for every open cover $\mathscr{U}$ of X, there is a sequence $\{\mathscr{V}_n\}$ of open refinements of $\mathscr{U}$ such that for each $x \in X$ there is some $n \in \omega$ such that $\bigcap \mathscr{V}_n(x)$ is a neighborhood of x in X.

It is clear that both submetacompact spaces and orthocompact spaces are suborthocompact. Note that each closed subspace of a suborthocompact space is suborthocompact.

Remark 1.3. Submetacompactness is preserved under closed maps (cf. [Ju2]). However, orthocompactness is not preserved under perfect maps. In fact, Burke [B1] gave an example of an orthocompact space X and a perfect map $f: X \to Y$ onto a nonorthocompact space Y. Seeing his proof for the nonorthocompactness of Y, it is easy to observe that the space Y is not suborthocompact. Hence suborthocompactness is not preserved under perfect maps.

2. Generalized Main Theorem

Let us start with the following lemma, which is easy to verify (e.g., see [B2, Theorem 6.1]).

Lemma 2.1. *If X is a submetacompact space and C is a compact Hausdorff space, then the product $X \times C$ is submetacompact.*

It has been proved in [GY] that the above "compact Hausdorff space" can be extended to "regular space with a σ-closure-preserving cover by compact sets."

Let X and Y be spaces. For a collection $\mathscr{H}$ of subsets of the product $X \times Y$ and an $(x, y) \in X \times Y$, let $\mathscr{H}(x, y) = \{H \in \mathscr{H} : (x, y) \in H\}$. A subset R of $X \times Y$ is an *open rectangle* if it is an open set of the form $A \times B$ (i.e., $R = A \times B$ such that A and B are open in X and Y, respectively).

In order to prove our Main Theorem, we are to prove the following generalized result, which is an analogue of [T2, Theorem 3.1], [M1, Theorem 2.7], and [Ju1, Theorem 4.1].

Theorem 2.2. *Let γX be a compactification of a Tychonoff space X. Then X is submetacompact if and only if the product $X \times \gamma X$ is suborthocompact.*

Proof. Since the "only if" part is an immediate consequence of Lemma 2.1, we prove only the "if" part.

Let $\mathscr{U}$ be any open cover of X. Take an open set U^* in γX with $U^* \cap X = U$ for each $U \in \mathscr{U}$. Let $\mathscr{U}^* = \{U^* : U \in \mathscr{U}\}$. Take a collection $\mathscr{V}$ of open sets in γX such that $\bigcup \mathscr{V} = \bigcup \mathscr{U}^*$ and, for each $V \in \mathscr{V}$, one can find $U_V \in \mathscr{U}$ with $\mathrm{Cl}_{\gamma X} V \subset U_V^*$. Let

$$\mathscr{G} = \{(V \cap X) \times U_V^* : V \in \mathscr{V}\} \cup \{(V \cap X) \times (\gamma X \setminus \mathrm{Cl}_{\gamma X} V) : V \in \mathscr{V}\}.$$

Then $\mathscr{G}$ is an open cover of $X \times \gamma X$. There is a sequence $\{\mathscr{H}_n\}$ of open refinements of $\mathscr{G}$ such that, for each $(x, x') \in X \times \gamma X$, one can choose $n \in \omega$ such that $\bigcap \mathscr{H}_n(x, x')$ is a neighborhood of (x, x') in $X \times \gamma X$. For each $(x, x') \in X \times \gamma X$ and each $n \in \omega$, we take an open rectangle $P_{x,x',n} \times Q_{x,x',n}$ in $X \times \gamma X$, that contains the point (x, x') and is contained in some member of $\mathscr{H}_n$. Since γX is compact, for each $x \in X$ and each $n \in \omega$, there are an open neighborhood $W_{x,n}$ of x in X and a finite subset $F(x, n)$ of γX such that

(i) $W_{x,n} \subset \bigcap_{z \in F(x,n)} P_{x,z,n}$,
(ii) $\gamma X = \bigcup_{z \in F(x,n)} Q_{x,z,n}$, and
(iii) $W_{x,n+1} \subset W_{x,n}$.

Here, we set $\mathscr{W}_n = \{W_{x,n} : x \in X\}$ for each $n \in \omega$. We show that the sequence $\{\mathscr{W}_n\}$ of open covers of X satisfies the condition of Theorem 1.1. For this, we need to prove the following claim, corresponding to a similar statement in the proof of [B2, Theorem 2.10].

Claim. For each $x \in X$, there is some $k \in \omega$ such that $\mathrm{Cl}_{\gamma X} \mathrm{St}(x, \mathscr{W}_k) \subset \mathrm{St}(x, \mathscr{U}^*)$.

Proof. Assume that we can pick some $x \in X$ such that

$$\mathrm{Cl}_{\gamma X} \mathrm{St}(x, \mathscr{W}_n) \setminus \mathrm{St}(x, \mathscr{U}^*) \neq \varnothing$$

for each $n \in \omega$. By (iii), $\mathrm{Cl}_{\gamma X}\,\mathrm{St}(x, \mathscr{W}_{n+1}) \subset \mathrm{Cl}_{\gamma X}\,\mathrm{St}(x, \mathscr{W}_n)$ for each $n \in \omega$. So there is some $q \in \bigcap_{n\in\omega} \mathrm{Cl}_{\gamma X}\,\mathrm{St}(x, \mathscr{W}_n) \setminus \mathrm{St}(x, \mathscr{U}^*)$. Choose $k \in \omega$ such that $\bigcap \mathscr{H}_k(x, q)$ is a neighborhood of (x, q) in $X \times \gamma X$. Take an open rectangle $S \times T$ in $X \times \gamma X$ such that $(x, q) \in S \times T \subset \bigcap \mathscr{H}_k(x, q)$. We can pick $p \in X$ such that $p \in T \cap \mathrm{St}(x, \mathscr{W}_k)$, and we can pick $y \in X$ such that $x \in W_{y,k}$ and $p \in W_{y,k}$. By (ii), pick $z \in F(y, k)$ with $q \in Q_{y,z,k}$. Then it follows from (i) that

$$\{(x, q), (p, q)\} \subset W_{y,k} \times Q_{y,z,k} \subset P_{y,z,k} \times Q_{y,z,k} \subset H_0,$$

for some $H_0 \in \mathscr{H}_k$. By $H_0 \in \mathscr{H}_k(x, q)$, we have

$$(x, p) \in S \times T \subset \bigcap \mathscr{H}_k(x, q) \subset H_0.$$

Hence we obtain the result that $\{(x, q), (x, p), (p, q)\} \subset H_0 \in \mathscr{H}_k$. By $q \notin \mathrm{St}(x, \mathscr{U}^*)$, note that $(x, q) \notin (V \cap X) \times U_V^*$ for each $V \in \mathscr{V}$. Since $\mathscr{H}_k$ refines $\mathscr{G}$, there is a $V_0 \in \mathscr{V}$ such that $H_0 \subset (V_0 \cap X) \times (\gamma X \setminus \mathrm{Cl}_{\gamma X} V_0)$. Therefore, we have

$$\{(x, p), (p, q)\} \subset (V_0 \cap X) \times (\gamma X \setminus \mathrm{Cl}_{\gamma X} V_0).$$

This is a contradiction. Our claim has been proved.

Pick any $x \in X$. By the claim, choose some $k \in \omega$ such that $\mathrm{Cl}_{\gamma X}\,\mathrm{St}(x, \mathscr{W}_k) \subset \mathrm{St}(x, \mathscr{U}^*)$. There is a finite subcollection $\mathscr{U}_x = \{U_1, \dots, U_m\}$ of $\mathscr{U}$ such that $\mathrm{Cl}_{\gamma X}\,\mathrm{St}(x, \mathscr{W}_k) \subset \bigcup_{i\le m} U_i^*$ and $x \in \bigcap_{i\le m} U_i^*$. Then we have $\mathrm{St}(x, \mathscr{W}_k) \subset \bigcup \mathscr{U}_x$ and $x \in \bigcap \mathscr{U}_x$. Thus it follows from Theorem 1.1 that X is submetacompact. The proof is complete.

Remark 2.3. In Theorem 2.2, a compactification γX of X can be replaced by a compact Hausdroff space K which contains X as a subspace, since suborthocompactness is inherited by closed subspaces.

For a space X, $L(X)$ denotes the Lindelöf number of X; i.e.,

$$L(X) = \aleph_0 \cdot \min\{\mathbf{m}\colon \text{ every open cover of } X \text{ has a subcover of cardinality} \le \mathbf{m}\}.$$

For a cardinality $\mathbf{m}$, $2^{\mathbf{m}}$ denotes the product of $\mathbf{m}$ copies of the discrete two-point space.

Morita [M2] actually showed that a Hausdroff space X is paracompact if and only if the product $X \times 2^{L(X)}$ is normal. As an analogue, Scott [S] showed that a space X is metacompact if and only if the product $X \times 2^{L(X)}$ is orthocompact. Here, we can also obtain the following analogue:

Theorem 2.4. *A space X is submetacompact if and only if the product $X \times 2^{L(X)}$ is suborthocompact.*

However, the proof is similar to that of [S, Lemma 1.6]. The details of it are left to the reader.

3. Katuta-Junnila's problem

A cover $\mathscr{V}$ of a space X is *directed* if, for each V_0, $V_1 \in \mathscr{V}$, there is some $V_2 \in \mathscr{V}$ such that $V_0 \cup V_1 \subset V_2$.

This problem, which was raised in [Ju2 and K], asks whether a space X is metacompact (submetacompact) if every directed open cover of X has a (σ-)cushioned refinement. Jiang [J] gave an affirmative answer to the metacompact case assuming that the space X is orthocompact. We are to generalize this result, using the suborthocompactness of X.

Theorem 3.1. *A space X is metacompact if and only if X is suborthocompact and every directed open cover of X has a cushioned refinement.*

Proof. Since the "only if" part is obvious from [Ju1, Theorem 3.1], we show only the "if" part.

Let $\mathscr{U}$ be an open cover of X. There is a sequence $\{\mathscr{V}_n\}$ of open refinements of $\mathscr{U}$ such that, for each $x \in X$, one can choose $n \in \omega$ such that $\bigcap \mathscr{V}_n(x)$ is a neighborhood of x in X. Take an $n \in \omega$. Let

$$X_n = \left\{x \in X\colon \bigcap \mathscr{V}_n(x) \text{ is a neighborhood of } x \text{ in } X\right\},$$

and let $\mathscr{V}_n^F$ be the open cover of X which consists of all finite unions of members of $\mathscr{V}_n$. Since $\mathscr{V}_n^F$ is directed, it has a cushioned refinement $\mathscr{C}_n$. So $\mathscr{V}_n^F$ has a subcover $\{V^*(C) : C \in \mathscr{C}_n\}$ such that $\mathrm{Cl}(\bigcup\{C\colon C \in \mathscr{A}\}) \subset \bigcup\{V^*(C)\colon C \in \mathscr{A}\}$ for each $\mathscr{A} \subset \mathscr{C}_n$. For each $x \in X_n$, let

$$W_{x,n} = \mathrm{Int}\left(\bigcap \mathscr{V}_n(x)\right) \setminus \mathrm{Cl}\left(\bigcup\{C \in \mathscr{C}_n\colon x \notin V^*(C)\}\right).$$

Then each $W_{x,n}$ is an open neighborhood of x in X.

Let $G_n = \bigcup\{W_{x,n}\colon x \in X_n\}$. Since every countable, increasing open cover of X is directed, it has a cushioned refinement. So it follows from [I, Corollary] that X is countably metacompact. Since $\{G_n\colon n \in \omega\}$ is a countable open cover of X, there is a point-finite open cover $\{H_n\colon n \in \omega\}$ of X such that $H_n \subset G_n$ for each $n \in \omega$. Here, we set

$$\mathscr{W} = \{W_{x,n} \cap H_n\colon x \in X_n \text{ and } n \in \omega\}.$$

Then $\mathscr{W}$ is an open refinement of $\mathscr{U}$. We show that $\mathscr{W}$ is a pointwise W-refinement of $\mathscr{U}$ (that is, for each $x \in X$, there is a finite subcollection $\mathscr{U}_x$ of $\mathscr{U}$ such that each member of $\mathscr{W}(x)$ is contained in some member of $\mathscr{U}_x$).

Pick $x \in X$. Let $N_x = \{n \in \omega\colon x \in H_n\}$. For each $n \in \omega$, we choose a $C_n \in \mathscr{C}_n$ and a finite subcollection $\mathscr{V}_{n,x}$ of $\mathscr{V}_n$ such that $x \in C_n$ and $V^*(C_n) = \bigcup \mathscr{V}_{n,x}$. We set $\mathscr{V}_x = \bigcup\{\mathscr{V}_{n,x}\colon n \in N_x\}$. Since N_x is finite, so is $\mathscr{V}_x$. Since each $\mathscr{V}_n$ refines $\mathscr{U}$, it suffices to show that each member of $\mathscr{W}(x)$ is contained in some member of $\mathscr{V}_x$. Assume that $x \in W_{y,n} \cap H_n$, where

$y \in X_n$. Then n is in N_x. If $y \notin V^*(C_n)$, then we have $W_{y,n} \cap C_n = \varnothing$, which contradicts $x \in C_n$. Hence we get $y \in V^*(C_n) = \bigcup \mathscr{V}_{n,x}$. Find some $V_0 \in \mathscr{V}_{n,x} \subset \mathscr{V}_x$ with $y \in V_0$. Then we have $W_{y,n} \subset \bigcap \mathscr{V}_n(y) \subset V_0$. Hence $\mathscr{W}$ is a pointwise W-refinement of $\mathscr{U}$. It follows from [W, Theorem 1] that X is metacompact. The proof is complete.

However, we cannot obtain an analogue of Theorem 3.1 for submetacompactness. So we raise it as a problem:

Problem 3.2. If every directed open cover of a suborthocompact space X has a σ-cushioned refinement, is X then submetacompact?

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