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THE SOUL AT INFINITY IN DIMENSION 4

GERARD WALSCHAP

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ABSTRACT. It is shown that 4-dimensional open manifolds with nonnegative sectional curvature whose fibers are totally geodesic are metrically rigid. For arbitrary dimension, one also concludes that the curvature is in a sense maximal at the soul.

1. INTRODUCTION

Let (M^n, g) be an open n-manifold with nonnegative sectional curvature K and soul S. It is known that there exists a Riemannian submersion $M \to S$ whenever $n \leq 4$ or the codimension of S is ≤ 2. In this note, we show that if the fibers of $M \to S$ are totally geodesic, then (M, g) itself is the result of a Riemannian submersion from a product:

Theorem 1. *Let M^n be an open n-manifold of nonnegative curvature with soul S, where $n \leq 4$ or $\operatorname{codim} S \leq 2$. If the fibers of $M \to S$ are totally geodesic, then there exists a Riemannian submersion $N \times P^k \to M$. Here N is compact, P^k is diffeomorphic to $\mathbb{R}^k$, both factors have metrics of nonnegative curvature, and $N \times P^k$ has the product metric.*

Roughly speaking, N is the soul of the Hausdorff limit $\lim_{t\to\infty}(M, \gamma(t))$, where γ is a ray from S. The proof of Theorem 1 relies heavily on the existence of a Riemannian submersion $M \to S$. It is not hard to see, however, that one always has an infinitesimal submersion along rays from S. In fact, we have the following weak version of O'Neill's formula for curvature:

Theorem 2. *Let M^n be an open n-manifold with nonnegative curvature K and soul S. Let $\gamma : [0, \infty) \to M$ be a ray starting from S, and P_t be the parallel translate along $\gamma|_{[0,t]}$ of some 2-plane $P_0 \subset T_{\gamma(0)}S$. Then the function $t \mapsto K_{P_t}$ is nonincreasing on $[0, \infty)$.*

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2. Proof of Theorem 1

The reader is referred to [2] for the general facts concerning nonnegative curvature, to [6, 7] for the $\operatorname{codim} S = 2$ case, and to [1] or [4] for Riemannian submersions. We shall discuss the two cases (namely $\dim M \leq 4$, and $\dim M > 4$ but $\operatorname{codim} S \leq 2$) simultaneously. Notice first that M may be assumed to be simply connected with soul of codimension 2. For if $\dim M \leq 4$, then the universal covering map $\tilde{M} \to M$ is a Riemannian submersion, and since these submersions are preserved under composition, we need only establish the theorem for $\tilde{M}$. But the soul $\tilde{S}$ of $\tilde{M}$ must have codimension 1 or 2, and in the first case, $\tilde{M}$ splits isometrically as $\tilde{S} \times \mathbb{R}$. Next, suppose $\dim M > 4$ and $\operatorname{codim} S \leq 2$. Consider once again the universal covering $\rho : \tilde{M} \to M$, and let $\tilde{S} = \rho^{-1}(S)$. By [2] $\tilde{S} = S_0 \times \mathbb{R}^k$, with S_0 compact, simply connected, and since $\tilde{S}$ is totally convex, there is a corresponding splitting $\tilde{M} = M_0 \times \mathbb{R}^k$. Then S_0 is a soul of M_0, of codimension $= \operatorname{codim} S$, which by the above may be assumed to equal 2, thereby establishing the claim.

Finally, we may assume that the normal bundle $\nu(S)$ of S in M is not flat, for otherwise, $M = S \times P^2$ isometrically. It follows that every geodesic orthogonal to S is a ray, and the exponential map $\exp_\nu : \nu(S) \to M$ is a diffeomorphism, cf. [6, 7]. Let

$$S(r) = \{q \in M / d(q, S) = r\}$$

denote the distance sphere of radius r around S with its metric g_r induced from M.

2.1. **Lemma.** *$(S(r), g_r)$ converges as $r \to \infty$ to (N, g_∞), where N is diffeomorphic to the unit sphere bundle $\nu^1(S)$, and g_∞ is a C^∞ metric of nonnegative curvature.*

Proof. Given $u \in \nu(S)$, let Ψ_u denote the canonical vector space isomorphism between the fiber through u and its tangent space at u. One has the polar coordinate vector field ∂_θ on $M \backslash S$ given by $\partial_{\theta|_{\exp u}} = \exp_* \Psi_u J_u$, where J is the complex structure on $\nu(S)$—recall that S is simply connected. Since the fibers of $M \to S$ are totally geodesic, ∂_θ is a Killing field on M, cf. [6, Lemma 1.7]. In particular, $G := |\partial_\theta|$ is a function which depends only on the distance from the soul. Moreover, $G : [0, \infty) \to \mathbb{R}$ is a concave, increasing, bounded function [6]. Set $\alpha := \lim_{r\to\infty} G(r) < \infty$. Notice that $(S(r), g_r)$, being convex, has nonnegative curvature by the Gauss equations, and the projection $S(r) \to S$ is a Riemannian submersion with totally geodesic fiber S^1 generated by ∂_θ. Define $\varphi_r : \nu^1(S) \to S(r)$ by $\varphi_r(v) = \exp(rv)$. Then $\varphi_r^* g_r$ is just the standard connection metric with $|\partial_\theta|$ rescaled by $G(r)$. Since $G(r) \to \alpha < \infty$, the lemma follows. □

2.2. *Remark.* (N, g_∞) can be viewed as the soul at infinity in the following sense: let $\gamma : [0, \infty) \to M$ be a ray from S. Then the Hausdorff limit $\lim^H_{t\to\infty} (M, \gamma(t)) = (X, 0)$, where X is isometric to $(N, g_\infty) \times \mathbb{R}$.

Resuming the proof of Theorem 1, recall that since $\nu(S)$ is a complex line bundle, $N \approx \nu^1(S)$ is just the corresponding principal S^1-bundle, and M is diffeomorphic to $(N \times \mathbb{R}^2)/S^1$. Endow $\mathbb{R}^2$ with the metric $\breve{g} = dr^2 + f^2(r)\, d\theta^2$, where $f^2(r) = \alpha^2 G^2(r)/(\alpha^2 - G^2(r))$. Then the diagonal action of S^1 on $(N, g_\infty) \times (\mathbb{R}^2, \breve{g})$ is by isometries. We claim that M is isometric to $(N \times \mathbb{R}^2)/S^1$. To see this, notice that $(N \times \mathbb{R}^2)/S^1$ is topologically $[0, \infty) \times N/\simeq$, where the fibers of N are collapsed at 0, and the metric has the form $dr^2 + d\sigma_r^2$, where $d\sigma_r^2$ is a metric on N obtained by rescaling in the fiber direction only. It thus suffices to check that the respective Killing fields of M and $(N \times \mathbb{R}^2)/S^1$ have the same norm. Now, if ρ denotes the Riemannian submersion $\rho : N \times \mathbb{R}^2 \to (N \times \mathbb{R}^2)/S^1$, then pointwise, the Killing field on the quotient is $\rho_*(0, \partial/\partial\theta)$, where $\partial/\partial\theta$ is the polar coordinate vector field on $\mathbb{R}^2$. Since the vertical space of ρ_* is spanned by $(\partial_\theta, -\partial/\partial\theta)$, one computes, for the horizontal component $(0, \partial/\partial\theta)^h$:

$$\begin{aligned}\left(0, \frac{\partial}{\partial\theta}\right)^h &= \left(0, \frac{\partial}{\partial\theta}\right) - \left\langle \left(0, \frac{\partial}{\partial\theta}\right), \left(\partial_\theta, -\frac{\partial}{\partial\theta}\right)\right\rangle \frac{1}{\alpha^2 + f^2} \left(\partial_\theta, -\frac{\partial}{\partial\theta}\right) \\ &= \frac{1}{\alpha^2 + f^2} \left(f^2 \partial_\theta, \alpha^2 \frac{\partial}{\partial\theta}\right).\end{aligned}$$

Thus

$$\left|\rho_* \left(0, \frac{\partial}{\partial\theta}\right)\right| = \left|\left(0, \frac{\partial}{\partial\theta}\right)^h\right| = \frac{\alpha f}{(\alpha^2 + f^2)^{1/2}} = G. \qquad \square$$

2.3. *Remark.* It is sometimes possible to obtain M from a Riemannian submersion even when the fibers of $M \to S$ are not totally geodesic. One such example is $M = S^2 \times \mathbb{R}^2$. Here, the metric on M is not a product metric, but comes from a submersion $S^2 \times \mathbb{R}^3 \to M$. On the other hand, consider $S^3 \times \mathbb{R}^2$ with the standard metric, and let S^1 act diagonally by isometries, on S^3 via the Hopf fibration, and on $\mathbb{R}^2$ by rotations. Then $M := (S^3 \times \mathbb{R}^2)/S^1 \to S = S^3/S^1 \approx S^2$ has totally geodesic fibers. Nevertheless, there exists a deformation g_ε of the metric g on M (with nonnegative sectional curvature), such that (M, g_ε) is *not* isometrically a quotient $(S^3 \times \mathbb{R}^2)/S^1$ for *any* metrics on S^3 and $\mathbb{R}^2$. Of course, the fibers of $M \to S$ are no longer totally geodesic. For details of both examples, see [6].

3. Proof of Theorem 2

We briefly recall some well-known facts about convex sets in a Riemannian manifold M of nonnegative sectional curvature. The reader should consult [2, 5] for proofs and further details. If $C \subset M$ is convex with nonempty boundary ∂C, and $\rho : C \to \mathbb{R}$, $\rho(p) := d(p, \partial C)$, denotes the distance function to the boundary, then $C^a := \{q \in C/\rho(q) \geq a\}$ is again convex. Moreover, if $C^a \neq \emptyset$, there exists a deformation retraction $\varphi : C \times [0, a] \to C^a$ which is distance nonincreasing. In general, the curves $t \mapsto \varphi(p, t)$, which can be regarded as integral curves of a generalized gradient $\nabla\rho$, need not be smooth. It is, however, easy to check that by the very construction of φ (see [5]) given $q \in C^a$, and a minimal geodesic γ from q to ∂C, $-\gamma$ is, up to reparametrization, an integral curve of $\nabla\rho$.

3.1. **Lemma.** *Let S be a soul of M, and let γ_i be rays with $\gamma_i(0) \in S$, $i = 1, 2$. Then the function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is nondecreasing.*

Proof. Performing the first step in the soul construction starting at $\gamma_1(0)$, we have, with the notation of [2], a family $\{C_t\}_{t\geq 0}$ of totally convex sets, with

$$S \subset C_0 = \{q \in C_t / d(q, \partial C_t) = t\}.$$

Fix any $t > 0$, and let ρ denote the distance function from ∂C_t. By definition, any ray from $\gamma_1(0)$ is a minimal connection to ∂C_t. But if $u \in T_{\gamma_1(0)}M$ denotes the parallel translate of $\dot{\gamma}_2(0)$ along some geodesic from $\gamma_2(0)$ to $\gamma_1(0)$, then $t \mapsto \exp(tu)$ is a ray (cf. [6]). It follows from [2, Theorem 1.10] that γ_2 itself is a minimal connection to ∂C_t. Thus both γ_1 and γ_2 are, up to negative reparametrization, integral curves of $\nabla\rho$. This establishes the lemma.

To complete the proof of Theorem 2, let x, y be an orthonormal basis of P_0, X, Y the parallel vector fields along γ with $X(0) = x$, $Y(0) = y$. For $t \geq 0$, $r > 0$, consider the circles $C_{r,t} : [0, 2\pi] \to M$,

$$C_{r,t}(\theta) = \exp_{\gamma(t)} r(\cos\theta \cdot X(t) + \sin\theta \cdot Y(t)).$$

By [2, Theorem 1.10], the minimal geodesic from $C_{r,0}(\theta)$ to $C_{r,t}(\theta)$ is a ray: indeed its initial tangent vector is the parallel translate of $\dot{\gamma}(0)$ along $s \mapsto \exp_{\gamma(0)} s(\cos\theta \cdot x + \sin\theta \cdot y)$. By 3.1, we have

$$d(C_{r,t}(\theta), C_{r,t}(\theta')) \leq d(C_{r,t'}(\theta), C_{r,t'}(\theta'))$$

for all θ, θ', $t \leq t'$. Thus the length of $C_{r,t}$ is not bigger than the length of $C_{r,t'}$, if $t \leq t'$. The theorem now follows from a well-known formula relating the length of $C_{r,t}$ as $r \to 0$ with the sectional curvature of P_t, see for example [3, p. 124]. □

3.2. *Question.* Theorem 2 says grosso modo that the curvature is maximal at the soul. Notice that $t \mapsto K_{P_t}$ need not be strictly decreasing, as in the case when $M = S \times P^k$ isometrically. More generally, suppose $R(x, y)\dot{\gamma}(0) = 0$. When the projection $M \to S$ is a Riemannian submersion, it is not hard to check that K_{P_t} is constant. Is this still true in general?

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Department of Mathematics, University of California, Los Angeles, California 90024