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POINTS NOT AS HYPERPLANE SECTIONS OF PROJECTIVELY NORMAL CURVES

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ABSTRACT. Here we show for many n, d (with $n \geq 6$) that the general set formed by d points in $\mathbf{P}^n$ is not the hyperplane section of an integral projectively normal curve in $\mathbf{P}^{n+1}$.

Here we give a negative answer to a part of a question raised in [EEK, p.53] proving the following theorem:

Theorem A. *Fix an integer $n \geq 6$. Then there is an integer d such that, for a general set S of d points in $\mathbf{P}^n$, there is no arithmetically Cohen-Macaulay integral locally complete intersection curve $C \subset \mathbf{P}^n$ with S as one of its hyperplane sections; more precisely, this occurs for all n, d satisfying the following two inequalities:*

$$d \leq (n+2)(n+1)/2, \tag{1}$$

$$(n-4)d > (n-2)(n+2) \tag{2}$$

and for all n, d such that there is an integer m with (n, d, m) satisfying the following conditions:

$$m \geq 3, \qquad n \geq 4m+4, \quad \text{and} \quad \binom{n+m-1}{n} < d \leq \binom{n+m}{n}. \tag{3}$$

Note that an affirmative answer to a related question (without the assumption "Cohen-Macaulay" in the statement of Theorem A) was given in [BM]. The proof of Theorem A will be just a dimensional count; however, the statement itself is interesting, and several mathematicians have asked about it. It is known to be false if $n = 2$ [CO].

Proof of Theorem A. Fix integers n, d satisfying (1) and (2) with $n \geq 2$ (hence $n \geq 6$), and consider $\mathbf{P}^n$ as a hyperplane H of $\mathbf{P} = \mathbf{P}^{n+1}$. The set of d points in H has dimension nd. Fix any set S of d points imposing for

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all $t \geq 2$ independent conditions on the forms of degree t in H (i.e., such that, in standard notation, $h^1(H, \mathbf{I}_{S,H}(t)) = 0$ for every $t \geq 2$); in particular, we may take as S a general set of d points of H. Assume the existence of an arithmetically Cohen-Macaulay integral curve $C \subset \mathbf{P}^{n+1}$, $\deg(C) = d$, with $C \cap H = S$; let $\mathbf{I}_C$ be the ideal sheaf of C in $\mathbf{P}^{n+1}$. From the exact sequences

$$0 \to \mathbf{I}_C(t-1) \to \mathbf{I}_C(t) \to \mathbf{I}_{S,H}(t) \to 0 \tag{4}$$

and descending induction on t, we get $h^2(\mathbf{P}^{n+1}, \mathbf{I}_C(1)) = 0$; i.e., $h^1(C, \mathbf{O}_C(1)) = 0$. Thus, since C is linearly normal, its arithmetic genus g is $d-n-1$. Let N_C be the normal sheaf $(\mathbf{I}_C/\mathbf{I}_C^2)^*$ to C in $\mathbf{P}^{n+1}$. By the Euler's sequence of $T\mathbf{P}^{n+1}$, there is a map $\mathbf{t}$: $(n+2)\mathbf{O}_C^n(1) \to N_C$ with $\operatorname{Coker}(\mathbf{t})$ supported at the singular points of C, hence with $\dim(\operatorname{Supp}(\operatorname{Coker}(\mathbf{t}))) = 0$. Thus $h^1(N_C) = 0$. Hence $h^0(N_C) = \chi(N_C)$. Since $H^0(C, N_C)$ is the tangent space to the Hilbert scheme $\operatorname{Hilb}(\mathbf{P}^{n+1})$ at the point C, $h^0(N_C)$ is an upper bound for the irreducible components of $\operatorname{Hilb}(\mathbf{P}^{n+1})$ containing C. If C is a locally complete intersection, by [H, Chapter III, Theorem 7.1], we have $\omega_C = \omega_{\mathbf{P}} \otimes c_1(N_C)$. Thus $\chi(N_C) = (n+2)d + (n-2)(1-g)$. Putting $g = d-n-1$, we see that the set of intersections $\{T \cap H\}$ with T in an irreducible component of $\operatorname{Hilb}(\mathbf{P}^{n+1})$ containing C cannot contain an open dense set of the symmetric product $S^d(H)$ if (2) holds.

Now fix integers n, m, and d satisfying (3). Fix $S \subset H$ with $\operatorname{card}(S) = d$, and an integral locally complete intersection arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^{n+1}$ with $\deg(C) = d$ and $S = C \cap H$; now we assume $h^1(H, \mathbf{I}_{S,H}(z)) = 0$ for $z \geq m$ and $h^0(H, \mathbf{I}_{S,H}(t)) = 0$ if $t < m$; thus $h^1(H, \mathbf{I}_{S,H}(w)) = d - h^0(H, \mathbf{O}_H(w))$ if $0 \leq w < m$. Of course, by (3) these assumptions are satisfied if S is a general set of d points in H. Set $h^1 = h^1(\mathbf{O}_C(1))$. Now we have $g := p_a(C) = d + h^1 - 1 - n$ by Riemann-Roch. By the classical proof of Castelnuovo's bound for the genus [H2, Chapter 3] and the fact that C is arithmetically Cohen-Macaulay, we have

$$h^1 = \sum_{t=2}^{m-1} h^1(H, \mathbf{I}_{S,H}(t)). \tag{5}$$

Using the very rough bound $h^1(N_C) \leq (n+2)h^1$, which comes from the Euler's sequence and the finiteness of $\operatorname{Supp}(\operatorname{Coker}(\mathbf{t}))$, we get the last part of Theorem A. $\square$

Remark 1. In the proof of Theorem A, we used the assumption that the curve is a locally complete intersection only to compute $\chi(N_C)$. This computation depends only on a formal neighborhood of the singularities of C. Thus, by [K, Lemma 2.2.11, p. 95], it is sufficient to assume that every nonlocally complete intersection point of C has embedding dimension 3. Furthermore, if we fix a measure δ of the failure of this local computation at the nonlocally complete intersection points, we can find n, d as in (1) for which the general S is not a hyperplane section of an integral arithmetically Cohen-Macaulay curve with degree of failure of $\chi(N_C)$ at most δ if instead of (2) we assume $(2)'$, which is as (2) except for a term $+\delta$ on the right-hand side.

We conjecture that for every $n \geq 6$ there is a $d(n)$ such that, if $d \geq d(n)$, a general set of d points of $\mathbf{P}^n$ is not the hyperplane section of an integral locally complete intersection arithmetically Cohen-Macaulay curve in $\mathbf{P}^{n+1}$; furthermore, it should be possible to drop the assumption that the curve is a locally complete intersection.

We would like to use this occasion to point out that the fact that the monodromy group of the general hyperplane section S of an integral curve is big ([H1] and [R] if $\mathrm{char}(\mathbf{K}) > 0$) implies not only that S is in uniform position, but also that any two subsets of S with the same cardinality have isomorphic minimal free resolutions (i.e., the same Betti numbers). We would like to say that S is syzygetic uniform position if this happens. Since the minimal free resolution of any hyperplane section of an arithmetically Cohen-Macaulay curve has as Betti numbers the Betti numbers of a minimal free resolution of the curve, this happens also in any characteristic for every hyperplane section of an integral arithmetically Cohen-Macaulay curve. Furthermore, if $k \leq n$ is an integer, we say that S is an syzygetic uniform position up to step k if there is an isomorphism between the first k steps of the minimal free resolution of any two subsets with the same cardinality of S. Uniform position implies syzygetic uniform position for points in the plane by [CO], but not in general. The easiest example (the verification is left to reader) is for $n = 3$, $d = 8$; start with seven general points P_i, $0 \leq i \leq 6$, in $\mathbf{P}^3$ (hence with homogeneous ideal generated by 3 quadrics and 1 cubic, e.g., by [B]); let D be the rational smooth curve containing the points P_i with $i > 0$; take a general point P_7 on D; set $S = \{P_i\}_{0 \leq i \leq 7}$.

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