

## A REMARK ON A PAPER OF JI-SHOU

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**ABSTRACT.** We observe that a strictly singular operator is not necessarily condensing, so that the invariant subspace problem for strictly singular operators remains open.

In the paper "Invariant subspace of strictly singular operators," which appeared in Proc. Amer. Math. Soc. **108** (1990), 931–936, the author Ji-Shou presents Theorem 1, which asserts that if  $X$  is a Banach space, then every strictly singular operator in  $L(X)$  is condensing. This is false. Indeed, if  $T \in L(X)$  is strictly singular but not compact, then for  $\alpha$  sufficiently large,  $\alpha T$  is still strictly singular but is not condensing. Thus the claim that a strictly singular operator has an invariant subspace remains unproven.

Preceding the assertion of Theorem 1 is Lemma 3 which states that if  $T$  is a bounded linear operator on a normed linear space  $X$ , and  $D$  is a bounded subset of  $X$  such that  $T$  is 1-1 on  $D$  and  $\gamma(T(D)) \geq \gamma(D)$ , then  $\gamma(T(D_1)) \geq \gamma(D_1)$  for every subset  $D_1$  of  $D$ . Here  $\gamma(S)$  is the measure of noncompactness of a set  $S$ , that is,  $\gamma(S) = \inf\{d \mid \text{there is a finite number of sets } S_i, i = 1, 2, \dots, n, \text{ such that } S = \bigcup_{i=1}^n S_i \text{ and } \text{diam } S_i \leq d\}$ . In fact, the lemma is false as seen by the following counterexample.

Let  $T: l_2 \rightarrow l_2$  be defined by

$$T(x_1, x_2, \dots) = (x_1, \frac{1}{2}x_2, x_3, \frac{1}{4}x_4, \dots).$$

Take  $D$  to be the unit disc and  $D_1 = \{x \in D \mid x_{2n+1} = 0, n = 0, 1, \dots\}$ . Now  $TD_1$  is precompact since the operator  $T_1: l_2 \rightarrow l_2$  defined by  $T_1(x_1, x_2, \dots) = (0, \frac{1}{2}x_2, 0, \frac{1}{4}x_4, \dots)$  is compact. Hence  $\gamma(T(D_1)) = 0 < \gamma(D_1)$ . But  $\gamma(T(D)) \geq \gamma(D)$ . Indeed, let  $D_2 = \{x \in D \mid x_{2n} = 0, n = 1, 2, \dots\}$ . Since the map  $J: l_2 \rightarrow l_2$  defined by

$$J(\alpha_1, \alpha_2, \dots) = (\alpha_1, 0, \alpha_2, 0, \dots)$$

is an isometry and  $JD = D_2$ , we have

$$\gamma(T(D)) \geq \gamma(T(D_2)) = \gamma(D_2) = \gamma(JD) = \gamma(D).$$

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