PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 3, July 1991

ESTIMATES FOR INVERSES OF e^{int} IN SOME QUOTIENT ALGEBRAS OF A^+

ELHASSAN ZEROUALI

(Communicated by Paul S. Muhly)

ABSTRACT. We give estimates for the norm of e^{int} in A^+/I, where I is a closed ideal of A^+ without inner factor, provided that the hull of I satisfies suitable geometric conditions.

INTRODUCTION

Let E be a closed subset of the unit circle Γ of Lebesgue measure zero, let $d\mu = \frac{1}{2\pi} d\lambda$, where $d\lambda$ is the Lesbegue measure, and let d be the metric defined by $d(e^{i\theta}, e^{i\theta'}) = |\theta - \theta'|/2\pi(\theta, \theta' \in [0, 2\pi])$.

Let A be the Wiener algebra such that

$$A = \left\{ f \in \mathscr{C}(\Gamma) \text{ such that } \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < +\infty \right\},$$

where $\|f\| = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$, and let A^+ be the subalgebra of A of functions in A such that $\hat{f}(n) = 0$ for $n < 0$. We say that E is a ZA^+ set if there exists a nonzero function in A^+ vanishing on E; we denote by I_E^+ the ideal of a function in A^+ vanishing on E and by $A^+(E)$ the quotient algebra $A^+/I^+(E)$.

It is shown in [2] that if E is a ZA^+ set we have

$$\|e^{-int}\|_{A^+(E)} = 0(\exp \varepsilon\sqrt{n}) \quad \text{for every } \varepsilon > 0.$$

Kahane and Katznelson prove in [5] that, for every $\beta > 0$, there exists a closed set $E \subset \Gamma$ such that

$$\int_0^{2\pi} \operatorname{Log} \frac{1}{d(e^{it}, E)} \, dt < +\infty$$

and

$$\lim_{n\to+\infty} \inf \frac{(\operatorname{Log}(\|e^{-int}\|_{A^+(E)})(\operatorname{Log} n)^{1+\beta}}{n^{1/2}} > 0.$$

Received by the editors January 10, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46J15, 46J20; Secondary 42A99.

In this paper, we prove that, for $1 < p \leq 2$, if E satisfies

$$\int_0^{2\pi} \operatorname{Log}^p \frac{1}{d(e^{it}, E)}\, dt < +\infty, \tag{1}$$

then there exists a $c > 0$ such that

$$\|e^{-int}\|_{A^+(E)} = 0(\exp c.n^{1/(1+p)}).$$

If I is a closed ideal of A^+ and the only common inner factor of the functions in I is 1, then, if $h(I) \subset \Gamma$ and satisfies (1), where

$$h(I) = \{z \in \overline{D} \text{ such that } f(z) = 0 \text{ for every } f \in I\},$$

the same estimates hold. Let E be a ZA^+ set and set $w_n = \|e^{-int}\|_{A^+(E)}$; we write $E^c = \bigcup_{\nu\in\mathbb{N}}]\alpha_\nu, \beta_\nu[$ and we set $L_\nu = \mu(]\alpha_\nu, \beta_\nu[) = |\beta_\nu - \alpha_\nu|/2\pi$. Let φ and ψ be defined for $t \in [0, 1]$ by:

$\varphi(t) = m(E_t)$, where $E_t = \{e^{i\theta} \in \Gamma \text{ such that } d(e^{i\theta}, E) \leq t\}$,

$\psi(t) = \sum_{L_\nu \leq t} L_\nu$.

We denote by N_ε the smallest integer N such that there exists a collection of arcs $(I_\alpha)_{\alpha\leq N_\varepsilon}$ of measure ε centered on elements of E and satisfying $E \subset \bigcup_{\alpha\leq N_\varepsilon} I_\alpha$.

We begin by remarking that certain conditions on functions ψ, φ, and $N_\varepsilon(E)$ are equivalent to (1).

Proposition 1. *Let $p \geq 1$. The following are equivalent.*

(a) $a = \sum_{\nu\in\mathbb{N}} L_\nu \operatorname{Log}^p \frac{1}{L_\nu} < +\infty$,

(b) $b = \int_0^1 \frac{\varphi(t)}{t} \operatorname{Log}^{p-1} \frac{1}{t}\, dt < +\infty$,

(c) $c = \int_0^1 \frac{\psi(t)}{t} \operatorname{Log}^{p-1} \frac{1}{t}\, dt < +\infty$,

(d) $d = \int_0^{2\pi} \operatorname{Log}^p \frac{1}{d(e^{it}, E)}\, dt, < +\infty$.

For $p = 1$ see [3] and [6]. The proof is similar for $p > 1$.

We now give another proposition, due to Atzmon, to be used in the proof of Theorem 3.

Proposition 2. *Let $f \in A^+$ and $\lambda \in D = \{z \in \mathbb{C}/|z| < 1\}$. Let Φ be defined by $\Phi(f, \lambda)(z) = \frac{f(z)-f(\lambda)}{z-\lambda}$ for $z \neq \lambda$ and $\Phi(f, \lambda)(\lambda) = f'(\lambda)$. Then $\Phi(f, \lambda) \in A^+$ and, for $\lambda \in D$,*

$$\|\Phi(f, \lambda)\|_{A^+} \leq \frac{2\|f\|_{A^+}}{1 - |\lambda|}.$$

If, in addition, $f \in I^+(E)$, then

$$f(\lambda)(\pi(\alpha) - \lambda)^{-1} = -\pi(\Phi(f, \lambda)), \tag{2}$$

where π is the canonical map from A^+ onto $A^+/I^+(E)$.

Proof. Set Lemma 1 and [2, Example I.3]. Recall that

$$\|e^{int}\|_{A^+(E)} = \|\pi(\alpha)^n\|_{A^+/I^+(E)}, \qquad n \in \mathbb{Z}$$

where $\alpha: e^{i\theta} \to e^{i\theta}$ for $\theta \in [0, 2\pi]$.

If $f(\lambda) \neq 0$, we obtain from (2):

$$\|(\pi(\alpha) - \lambda)^{-1}\|_{A^+/I^+(E)} \leq \frac{1}{|f(\lambda)|} \frac{2\|f\|_{A^+}}{1 - |\lambda|}. \tag{3}$$

Theorem 3. *If E satisfies* (1), *for some $p \in]1, 2]$, then E is a ZA^+ set and there exists $c > 0$ such that*

$$\omega_n = 0(\exp c.n^{1/(1+p)}).$$

Proof. As in [3], we define h by

$$h(t) = K\left(\log\frac{2\pi}{t - \alpha_\nu} + \log\frac{2\pi}{\beta_\nu - t}\right), \qquad t \in]\alpha_\nu, \beta_\nu[, \ \nu \in \mathbb{N}.$$

Using Proposition 1, we see that $h \in L^p(\Gamma) \subset L^1(\Gamma)$, so the function

$$f(z) = \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(t)\, dt\right) \text{ is analytic in } D.$$

We have $f \in A^+$ for $K > 2$ [3], and

$$|f(e^{it})| = \lim_{r \to 1} |f(re^{it})| = \left\{\frac{(t - \alpha_\nu)(\beta_\nu - t)}{4\pi^2}\right\}^K.$$

Thus $f|_E = 0$, and E is a ZA^+ set.

Let

$$g(z) = \frac{1}{f(z)} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(t)\, dt\right) \qquad (z \in D).$$

We have

$$\log|g(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) h(t)\, dt \qquad (0 \leq r < 1).$$

Since $P_r(\theta - t) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta - t)}$, we obtain

$$\begin{aligned} \operatorname{Log}|g(re^{i\theta})| &= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta - t)} h(t)\right) dt \\ &= \frac{1}{2\pi}\sum_{n \in \mathbb{Z}} r^{|n|} \left(\int_0^{2\pi} e^{-int} h(t)\, dt\right) e^{in\theta} \\ &= \sum_{n \in \mathbb{Z}} r^{|n|} \hat{h}(n) e^{in\theta}. \end{aligned}$$

Recall that if $h \in L^p$ for $1 < p \leq 2$, then $(\hat{h}(n))_n \in l^q(\mathbb{Z})$ and $\{\sum_{n\in\mathbb{Z}} |\hat{h}(n)|^q\}^{1/q} \leq \|h\|_p$ where $\frac{1}{p} + \frac{1}{q} = 1$ [7, p. 98].

So we have

$$\begin{aligned}\log|g(re^{i\theta}) &\leq \left\{\sum_{n\in\mathbb{Z}} (r^{|n|})^p\right\}^{1/p} \|h\|_p \\ &\leq 2^{1/p} \frac{1}{(1-r)^{1/p}} \|h\|_p ,\end{aligned}$$

and by (2) we have

$$\|(\pi(\alpha) - \lambda)^{-1}\|_{A^+(E)} \leq \frac{2\|f\|_{A^+} \exp 2^{1/p} \|h\|_p (1-|\lambda|)^{-1/p}}{1-|\lambda|} .$$

Using Lemma 2 in [2] we obtain that there exists $b, c > 0$ such that

$$\|\pi(\alpha)^{-n}\| \leq b . \exp c.n^{1/(1+p)} \quad \text{for } n \geq 0 .$$

This proves the theorem.

We give an application of this method to some other closed ideals.

Let I be a closed ideal of A^+, and let

$$h(I) = \{z \in \overline{D} \text{ such that } f(z) = 0 \text{ for every } f \in I\} .$$

A consequence of Taylor's and Williams's estimates [8, Lemmas 5.8 and 5.9] is that, if $f \in A^+$ and $f(e^{it}) = 0(\text{dist}(e^{it}, h(I))^2$, and if the only common inner factor of elements of I is 1, then $f \in I$ [4].

Using this result we obtain:

Theorem 4. *Let I be a closed ideal of A^+ and let $\pi \colon A^+ \to A^+/I$ be the canonical map. If I is such that*:

(a) *$h(I)$ satisfies* (1) *for some* $p \in]1, 2]$, *and*

(b) *the only common inner factor of all elements of I is* 1,

then there exists $c > 0$ such that

$$\|\pi(\alpha)^{-n}\|_{A^+/I} = 0(\exp c.n^{1/(1+p)}) .$$

Proof. From (b) we have $h(I) \subset \Gamma$. Since $h(I)$ satisfies (1), if f is the function defined in theorem 3 for $E = h(I)$, then f verifies

$$f(e^{i\theta}) = 0(\text{dist}(e^{i\theta}, h(I)))^K , \qquad K > 2 .$$

Thus $f \in I$.

We see as above that

$$\|(\pi(\alpha) - \lambda)^{-1}\| \leq \frac{2\|f\|_{A^+}}{1-|\lambda|} ,$$

and the estimates for $\|\pi(\alpha)^{-n}\|$ follow from the same argument as in the proof of Theorem 3.

References

1. A. Atzmon, *Boundary values of absolutely convergent Taylor's series*, Ann. of Math. **111** (1980), 231–237.
2. ——, *Operators which are annihilated by analytic function and invariant subspaces*, Acta Math. **144** (1980), 27–63.
3. L. Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta Math. **87** (1952), 325–345.
4. J. Esterle, E. Strouse, and F. Zouakia, *Closed ideals in* A^+ (to appear).
5. J. P. Kahane, *Séries de Fourier absolument convergentes*, Springer-Verlag, New York, 1970.
6. J. P. Kahane and Y. Katznelson, *Sur les algèbres de restriction des séries de Fourier absolument convergentes à un fermé du cercle*, J. Anal. Math. **23** (1970), 185–197.
7. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968.
8. B. A. Taylor and D. I. Williams, *Ideals in rings of analytic functions with smooth boundary values*, Canad. J. Math. **22** (1970), 1266–1283.

Unité de Formation et de Recherche de Mathématiques, Université de Bordeaux I, 351 Cours de la Libération, 33405 Talence, France