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CARLESON MEASURES IN HARDY AND WEIGHTED BERGMAN SPACES OF POLYDISCS

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ABSTRACT. The importance of theorems on Carleson measures has been well recognized [3]. In [1] Chang has given a characterization of the bounded measures on $L^p(T^n)$ using what one may characterize as the bounded identity operators from Hardy spaces of polydiscs in L^p spaces. In [4] Hastings gives a similar result for (unweighted) Bergman spaces of polydiscs. In this paper we characterize the *bounded* identity operators from weighted Bergman spaces of polydiscs into L^p spaces, and classify those operators which are compact on the Hardy and weighted Bergman spaces in terms of Carleson-type conditions. We give two immediate applications of these results here, and a much broader class of applications elsewhere [5].

1. DEFINITIONS AND PRELIMINARIES

We denote by U^n the unit polydisc in C^n, by T^n the distinguished boundary of U^n, by H^p the Hardy space of order p in U^n, by A^p_α the weighted Bergman spaces of order p with weights $\prod_{i=1}^n(1-|z_i|^2)^\alpha$, $\alpha > -1$. D_α will be used to denote the weighted Dirichlet spaces with respect to these weights. We shall use m_n to denote the n-dimensional Lebesgue area measure on T^n, normalized so that $m_n(T^n) = 1$. By σ_n we shall denote the volume measure on $\overline{U^n}$, defined so that $\sigma_n(\overline{U^n}) = 1$, and by $\sigma_{n,\alpha}$ we shall denote the weighted measure on $\overline{U^n}$ given by $\prod_{i=1}^n(1-|z_i|^2)^\alpha\sigma_n$. We use R to describe rectangles on T^n, and we use $S(R)$ to denote the corona associated to these sets. In particular, if I is an interval on T of length δ centered at $e^{i(\theta_0+\delta/2)}$,

$$S(I) = \{z \in U: 1-\delta < r < 1,\ \theta_0 < \theta < \theta_0+\delta\}.$$

Then if $R = I_1 \times I_2 \times \cdots \times I_n \subset T^n$, with I_j intervals having length δ_j and having centers $e^{i(\theta_j^0+\delta_j/2)}$, $S(R)$ is given by $S(R) = S(I_1)\times S(I_2)\times\cdots\times S(I_n)$. We use S to denote $S(R)$ whenever convenient. If V is any open set in T^n we define $S(V) = \bigcup_\alpha S(R_\alpha)$ where $\{R_\alpha\}$ runs through all rectangles in V. A

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finite, nonnegative, Borel measure μ on $\overline{U^n}$ is said to be a (bounded) Carleson measure if $\mu(S(V)) \leq Cm_n(V)$ for all connected open sets $V \subset T^n$. μ is said to be a compact-Carleson measure if

$$\lim_{m_n(V)\to 0} \sup_{V\subset T^n} \frac{\mu(S(V))}{m_n(V)} = 0.$$

μ, or where appropriate μ_α, is said to be a (bounded) α-Carleson measure if $\mu(S(R)) \leq C\prod_{i=1}^n \delta_i^{2+\alpha}$ for all $R \subset T^n$, and likewise μ, or μ_α, is a compact α-Carleson measure if

$$\lim_{\delta_i\to 0} \sup_{\theta\in T^n} \frac{\mu(S(R))}{\prod_{i=1}^n \delta_i^{2+\alpha}} = 0.$$

We point out that Carleson measures as they are used in the literature are always bounded. One should also note that the definitions of these measures have nothing to do with Hardy or weighted Bergman spaces. The definition of Carleson measures for polydiscs is due to Chang [1].

By considering the homogeneous expansion of f it is possible to obtain the following characterizations of A_α^2 and D_α.

Proposition 1.1. *For* $\alpha > -1$

(i)

$$f \in A_\alpha^2(U^n) \Leftrightarrow \sum_{|s|=0}^{\infty} |a_s|^2 \prod_{i=1}^{n} (s_i+1)^{-1-\alpha} < \infty,$$

(ii)

$$f \in D_\alpha(U^n) \Leftrightarrow \sum_{|s|=0}^{\infty} |a_s|^2 \sum_{j=1}^{n} s_j^2 \prod_{i=1}^{n} (s_i+1)^{-1-\alpha} < \infty,$$

where $|s| = \sum_{i=1}^n s_i$ *and multi-index notation is used.*

An immediate consequence of this proposition and the definitions is

Corollary 1.2. *For* $0 < p < \infty$ *and* $-1 < \alpha < \beta$

(i) $D_\alpha(U^n) \subset A_\alpha^2(U^n)$,
(ii) $A_\alpha^p(U^n) \subset A_\beta^p(U^n)$,
(iii) $D_\alpha(U^n) \subset D_\beta(U^n)$,
(iv) $A_{-1}^2(U^n) = H^2(U^n)$, *and*
(v) $A_\alpha^2 \subset D_{\alpha+2}$.

The inclusions are strict.

We note that the equivalence that exists between weighted Bergman and other weighted Dirichlet spaces in the disc or the unit ball (see [8], for example) fails in polydiscs. In fact, to recover this equivalence we need to replace the weights by $(1-\|z\|^2)^{n\alpha}$ where $\|\cdot\|$ is the polydisc norm.

We shall show that whenever μ_α is a finite, positive, Borel measure on U^n of the type $\prod_{i=1}^n (1-|z_i|^2)^\alpha \, d\nu_n$, where ν_n is an arbitrary positive, Borel measure on U^n (not depending on α), then we have the following proposition.

Proposition 1.3. *Suppose* $-1 < \alpha < \beta$ *and let* I_α *be the identity operator from* $A^p_\alpha(U^n)$ *into* $L^p(\mu_\alpha)$. *If* I_α *is a bounded* (*compact*) *operator, then so is* I_β.

We shall prove this proposition at the end of the next section where we shall discuss the relationship between the operator I_α and a class of positive, finite, Borel measures on $\overline{U^n}$. In the proceeding, we shall frequently make use of the following two classical results [9, Example 18, p. 107]:

Lemma 1.4. (i) *Let* X *be a reflexive Banach space. Then an operator* $T: X \to X$ *is compact if and only if* T *takes weakly convergent sequences to norm convergent sequences in* X.

(ii) *In* H^p *and* A^p_α, $p > 1$, *weak convergence is equivalent to norm boundedness and pointwise convergence.*

This lemma states that compactness of an operator T on $H^p(U^n)$ or $A^p_\alpha(U^n)$ is equivalent to T taking every norm bounded sequence f_n converging to zero uniformly on compact subsets of U^n to a sequence converging to zero in the appropriate norm.

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2. Main results

Garnett [3] provides a long history of the development and application of Carleson measures. This rich area of research contains a large body of literature on characterizations of different classes of operators in different spaces and their applications. Chang [1] has characterized the bounded measures on $L^p(T^2)$ using a two-line proof referring to a result of Stein. Characterization of the bounded identity operators on Hardy spaces is an immediate consequence of Chang's proof using standard arguments. Hastings [4] has given a similar result for unweighted Bergman spaces. Recently MacCluer [7] has obtained a Carleson measure characterization of identity operators on Hardy spaces of the unit ball of C^n using the well-known results of Hormander, and Stegenga [10] has considered similar questions in the unit disc. In this work, which is inspired by the results of Chang and Hastings, we present three theorems which characterize the compact identity operators from Hardy spaces of polydiscs into L^p spaces, and the corresponding bounded and compact identity operators on weighted Bergman spaces. As an immediate application of these characterizations we prove Proposition 1.3 and a weighted Fejer-Riesz inequality. Further application of these results to a large class of operators on these spaces are discussed elsewhere [5]. For reference we shall state the result of Chang here.

Theorem 2.1 (Chang). *Let μ be a positive finite measure on $\overline{U^n}$. Then μ is bounded in $L^p(T^n)$, $1 < p < \infty$, i.e.*

$$\int_{\overline{U^n}} |P[f]|^p \, d\mu \le C \int_{T^n} |f|^p \, dm_n \tag{1}$$

if and only if

$$\mu(S(V)) \le C m_n(V) \quad \textit{for all connected open sets } V \subset T^n; \tag{2}$$

i.e., (1) *holds if and only if μ is a Carleson measure for H^p.*

$P[f]$ in (1) denotes the Poisson integral of the function f. In part, the significance of this theorem is explained by this corollary which follows by using a standard perturbation of the proof of Chang.

Corollary 2.2. *μ is bounded in $H^p(U^n)$, $1 < p < \infty$, if and only if μ is a Carleson measure.*

The proof of the converse direction of this theorem makes use of the results on nontangential convergence for Poisson integral and the Hardy-Littlewood maximal theorem which are valid for $1 < p < \infty$, and thus this proof does not extend to $p = 1$. An important consequence of this result is that the boundedness of the measure μ is evidently independent of the choice of p. A similar result holds between the compact Carleson measures on $H^p(U^n)$ and the compact identity operators from $H^p(U^n)$ into $L^p(\mu)$. Precisely,

Theorem 2.3. *Let μ be a finite, nonnegative, Borel measure on $\overline{U^n}$ and assume that $H^p(U^n) \subset L^p(\mu)$. Let A be the identity operator sending $H^p(U^n)$ to $L^p(\mu)$, $1 < p < \infty$. Then A is a compact operator if and only if μ is a compact Carleson measure for H^p, i.e. μ satisfies*

$$\lim_{\delta \to 0} \sup_{m_n(V) < \delta} \frac{\mu(S(V))}{m_n(V)} = 0. \tag{1}$$

The assumption $H^p(U^n) \subset L^p(\mu)$ is needed in order for the identity operator to send H^p into L^p.

Proof. First, suppose that A is compact, and to show a contradiction suppose that (1) fails. Then there is an $\varepsilon > 0$ and a sequence of open sets $V_j \subset T^n$ with $\lim_n m_n(V_j) = 0$ so that

$$\mu(S(V_j)) > \varepsilon m_n(V_j). \tag{2}$$

It is easy to show that there exists a sequence of rectangles $R_j \subset T^n$ so that $m_n(R_j) \to 0$ as $j \to \infty$ for which

$$\mu(S(R_j)) \ge \frac{\varepsilon}{2} m_n(R_j). \tag{3}$$

Say each R_j has its center at $(e^{i(\theta_1^0+\delta_1/2)}, e^{i(\theta_2^0+\delta_2/2)}, \dots, e^{i(\theta_n^0+\delta_n/2)})_j = \eta_j$. Set $\alpha_{j_k} = (1-\delta_{j_k})e^{i(\theta_j+\delta_j/2)_k}$, and define f_j on T^n so that

$$P[f_j](z_1, z_2, \dots, z_n) = \prod_{i=1}^{n} (1 - \overline{\alpha}_{ij} z_i)^{-4/p}.$$

Then

$$16^{-n}\prod_{i=1}^{n}\delta_i^{-4}\mu(S(R_j)) \le \int_{S(R_j)} |P[f_j]|^p\, d\mu \le \int_{\overline{U^n}} |P[f_j]|^p\, d\mu,$$

and $\int_{T^n} |f_j|^p\, dm_n \le \prod_{i=1}^{n}\delta_i^{-3}$. Define $g_j = P[f_j]/\|f_j\|_p$. Here $\|\cdot\|_p$ denotes the H^p norm of f_j. Then $g_j \in H^p(U^n)$ converges to zero weakly since $\delta_i \to 0$ as $j \to \infty$, and

$$\begin{aligned}\int_{\overline{U^n}} |g_j|^p\, d\mu &\ge 16^{-n}\prod_{i=1}^{n}\delta_i^{-4}\frac{\mu(S(R_j))}{\prod_{i=1}^{n}\delta_i^{-3}} = 16^{-n}\prod_{i=1}^{n}\delta_i^{-1}\mu(S(R_j))\\ &\ge 16^{-n}\frac{\varepsilon}{2}m_n(R_j)\prod_{i=1}^{n}\delta_i^{-1} = C\varepsilon.\end{aligned}$$

Hence, the sequence g_j satisfies $\int_{\overline{U^n}} |g_j|^p\, d\mu \ge C\varepsilon$. This contradicts the compactness of A.

Conversely, suppose that (1) holds; i.e., suppose that

$$\lim_{\delta\to 0}\ \sup_{m_n(V)\le\delta}\frac{\mu(S(V))}{m_n(V)} = 0. \tag{4}$$

Then given $\varepsilon > 0$, there exists a $\delta_0 > 0$ so that for all $\delta < \delta_0$, $V \subset T^n$ and $m_n(V) \le \delta$

$$\mu(S(V)) \le \varepsilon m_n(V). \tag{5}$$

We show that A is compact. By Lemma 1.4 it suffices to show that A takes bounded sequence f_j converging to zero pointwise in U^n to a sequence converging to zero in $L^p(\mu)$ norm. Decompose μ so that $\mu = \mu_1 + \mu_2$, where μ_1 is the restriction of μ to $(1-\delta_0)\overline{U^n}$, and $\mu_2 = \mu - \mu_1$. Then

$$\int_{\overline{U^n}} |f_j|^p\, d\mu = \int_{\overline{U^n}} |f_j|^p\, d\mu_1 + \int_{\overline{U^n}} |f_j|^p\, d\mu_2.$$

Since $\mu_2 < \mu$, μ_2 is a compact Carleson measure whenever μ is; i.e., μ_2 satisfies (5). We claim that μ_2 satisfies the Carleson condition (2) in Theorem 2.1 with C replaced by 2ε. To see this claim note that if $V \subset T^n$ so that $S(V) \subset N$, $N = \overline{U^n} - (1-\delta_0)\overline{U^n}$, the claim is immediate from (5). For an arbitrary $V \subset T^n$, decompose V into a union of open sets V_j so that $m_n(V_j) < \delta_0$ and $\sum_j m_n(V_j) \le 2m_n(V)$. Then $S(V_j) \subset N$, and $S(V) \cap N = \bigcup_{j=1}^{n} S(V_j)$. Hence

$$\begin{aligned}\mu_2(S(V)) &= \mu(S(V)\cap N)\\ &\le \sum_j \mu(S(V_j)) \le \varepsilon\sum_j m_n(V_j) \le 2\varepsilon m_n(V).\end{aligned}$$

Therefore $\mu_2(S(V)) \le 2\varepsilon m_n(V)$ for all $V \subset T^n$. Hence μ_2 satisfies the Carleson condition (2) in Theorem 2.1 with $C = 2\varepsilon$ and thus

$$\int_{\overline{U^n}} |f_j|^p\, d\mu_2 \le 2\varepsilon\int_{T^n} |f_j|^p\, dm_n \le 2M\varepsilon,$$

where the second inequality is because f_j is a norm bounded family. Hence the integral with respect to μ_2 may be made as small as required. But for δ_0 fixed, the integral with respect to μ_1 can also be made as small as required by choosing j sufficiently large. This is because f_j converge to zero uniformly on compact subsets of U^n. $\square$

This result shows that compactness of the operator A is independent of p.

We now consider an analogous theorem for the weighted Bergman spaces of polydiscs. For $A_0^2(U^n)$ Hastings' [4] results imply:

Theorem 2.4 (Hastings). *Let μ be a finite positive measure on $\overline{U^n}$. Then there exists a constant $C > 0$ so that for $1 < p < \infty$*

$$\int_{\overline{U^n}} |f(z_1, z_2, \dots, z_n)|^p \, d\mu \le C \int_{\overline{U^n}} |f(z_1, z_2, \dots, z_n)|^p \, d\sigma_n \tag{1}$$

for every $f \in A_0^p(U^n)$ if and only if there is a constant $C > 0$ so that

$$\mu(S(R)) \le C \prod_{i=1}^{n} \delta_i^2 \tag{2}$$

for every set $S(R)$ given by definition (i) *at the beginning of this section.* $\square$

Condition (2) states that μ is an 0-Carleson measure for $A_0^p(U^n)$. Stegenga [10] obtains a result of this nature for $A_\alpha^p(U)$ for arbitrary $\alpha > -1$. MacCluer and Shapiro [8] obtain compactness conditions for the weighted Bergman spaces of the disc. We extend Stegenga and MacCluer-Shapiro's results to polydiscs.

Theorem 2.5. *Let $I_\alpha : A_\alpha^p(U^n) \to L^p(\mu)$, $1 < p < \infty$, $\alpha > -1$ be the identity map. Then*

(i) *I_α is a bounded operator if and only if μ is an α-Carleson measure.*
(ii) *I_α is a compact operator if and only if μ is a compact α-Carleson measure.*

Proof. In (i) we show that if $f \in A_\alpha^p(U^n)$ then

$$\int_{\overline{U^n}} |f|^p \, d\mu \le C \int_{\overline{U^n}} |f|^p \, d\sigma_{n,\alpha} \tag{1}$$

if and only if

$$\mu(S(R)) \le C \prod_{i=1}^{n} \delta_i^{2+\alpha}. \tag{2}$$

To prove (2) suppose that (1) holds for all $f \in A_\alpha^p(U^n)$. Define

$$f(z) = \prod_{j=1}^{n} (1 - \overline{\alpha_j} z_j)^{-(\alpha+2)/p},$$

where $\alpha_j = (1-\delta_j)e^{i(\theta_j^0+\delta_j/2)}$. Since on $S(R)$ $|f(z)|^p > 2^{-(\alpha+2)}\prod_{j=1}^n \delta_j^{-(\alpha+2)}$,

$$\int_{\overline{U^n}} |f|^p\, d\mu \ge \int_{S(R)} |f|^p\, d\mu \ge 2^{-(\alpha+2)}\prod_{j=1}^n \delta_j^{-(\alpha+2)}\mu(S(R)). \tag{3}$$

On the other hand, $f(z)$ is clearly in $A_\alpha^p(U^n)$. Combining (1) and (3) we get $\mu(S(R)) \le C\prod_{j=1}^n \delta_j^{\alpha+2}$. Hence μ is an α-Carleson measure. Conversely, to prove (1) suppose that (2) holds for all rectangles in T^n. Fix $z \in U^n$ and let $1-|z_j|^2 = \delta_j$. Consider a polydisc W_z centered at z and radius $\delta_j/2$ in the z_j coordinate. If $R = I_1 \times \cdots \times I_n$ is the rectangle on T^n with I_j centered at $z_j/|z_j|$ and $|I_j| = 2\delta_j$, then $W_z \subset S(R)$. Thus for $f \in A_\alpha^p(U^n)$ the sub-mean-value property for $|f|$ gives

$$\begin{aligned}
|f(z)| &\le \frac{C}{\prod_{j=1}^n (1-|z_j|^2)^2}\int_{W_z} |f|\, d\sigma_n \\
&\le \frac{C}{\prod_{j=1}^n (1-|z_j|^2)^{\alpha+2}}\int_{W_z} |f|\prod_{j=1}^n (1-|w_j|^2)^\alpha\, d\sigma_n \\
&\le \frac{C}{\prod_{j=1}^n \delta_j^{\alpha+2}}\int_{S(R)} |f|\, d\sigma_{n,\alpha}.
\end{aligned}$$

Noting that $\sigma_{n,\alpha}(S(R)) = C\prod_{j=1}^n \delta_j^{2+\alpha}$, we have

$$|f(z)| \le \frac{C}{\sigma_{n,\alpha}(S(R))}\int_{S(R)} |f|\, d\sigma_{n,\alpha}. \tag{4}$$

This inequality is the heart of the proof. Now define

$$B_\alpha[f](z) = \sup_R \frac{1}{\sigma_{n,\alpha}(S(R))}\int_{S(R)} |f|\, d\sigma_{n,\alpha}.$$

We have

$$|f(z)| \le CB_\alpha[f](z). \tag{5}$$

In addition, by hypothesis $\mu(S(R)) \le C\sigma_{n,\alpha}(S(R))$. Using a natural generalization of the argument used for getting the weak type $(1, 1)$ inequality (for example, as in Theorem 9.4 [2]), we show that there exists a constant C independent of s so that

$$\mu\{B_\alpha[f] > s\} \le Cs^{-1}\|f\|_{1,\alpha}. \tag{6}$$

Given (6), since B_α is obviously a sublinear operator of type (∞, ∞), by the Marcinkiewicz interpolation theorem B_α is a bounded operator from $L^p(\sigma_{n,\alpha})$ into $L^p(\mu)$ for $p > 1$. Thus

$$\int_{\overline{U^n}} |B_\alpha[f]|^p\, d\mu \le C\int_{\overline{U^n}} |f|^p\, d\sigma_{n,\alpha}. \tag{7}$$

Using (5) and (7) we conclude

$$\begin{aligned}\int_{\overline{U^n}} |f|^p \, d\mu &\le C \int_{\overline{U^n}} |B_\alpha[f]|^p \, d\mu \quad \text{(by (5))} \\ &\le C \int_{\overline{U^n}} |f|^p \, d\sigma_{n,\alpha} = C\|f\|_{p,\alpha}^p.\end{aligned}$$

This proves (1). To complete the proof we must prove (6); i.e., we need to show that if $\mu(S(R)) \le C\sigma_{n,\alpha}(S(R))$ then $\mu\{B_\alpha[f] > s\} \le Cs^{-1}\|f\|_{1,\alpha}$. Let $R_z = I_1 \times \cdots \times I_n$ denote a rectangle on T^n with I_j denoting intervals centered at $z_j/|z_j|$ and radius $(1 - |z_j|)/2$. Let S_z denote the corona associated with R_z. Note that $z \in S_z$. Define

$$A_s^\varepsilon = \left\{ z \in U^n : \int_{S_z} |f| \, d\sigma_{n,\alpha} > s(\varepsilon + \sigma_{n,\alpha}(S_z)) \right\}, \tag{8}$$

and note that $\Lambda = \{z \in U^n : B_\alpha[f] > s\} = \bigcup_{\varepsilon > 0} A_s^\varepsilon$, i.e. $\mu(\Lambda) = \lim_{\varepsilon \to 0} \mu(A_s^\varepsilon)$. Further note that if $z \in A_s^\varepsilon$ and S_z are disjoint for the different $z \in \Lambda$, then by (8)

$$s \sum_{z \in \Lambda} (\varepsilon + \sigma_{n,\alpha}(S_z)) < \sum_{z \in \Lambda} \int_{S_z} |f| \, d\sigma_{n,\alpha} \le \|f\|_{1,\alpha}.$$

Therefore we have

$$s \sum_{z \in \Lambda} (\varepsilon + \sigma_{n,\alpha}(S_z)) \le \|f\|_{1,\alpha}. \tag{9}$$

In particular, (9) shows that there are only finitely many $z \in A_s^\varepsilon$ so that their corresponding S_z are disjoint. From these extract the points, $z_1, \ldots, z_l$, that in addition have the property that if their associated S_z radii are multiplied by five in each coordinate the resulting sets cover A_s^ε. This follows from a standard covering lemma. Write the S_z associated with these points as $S_1, \ldots, S_l$. Since $A_s^\varepsilon \subset \bigcup_{k=1}^l 5S_k$, S_k are pairwise disjoint,

$$\mu(A_s^\varepsilon) \le 5^n \sum_{k=1}^l \mu(S_k). \tag{10}$$

Also, since by hypothesis

$$\mu(S_z) \le C\sigma_{n,\alpha}(S_k), \tag{11}$$

combining (10), (11), and (9) gives

$$\begin{aligned} s\mu(A_z^\varepsilon) &\le 5^n s \sum_{k=1}^l \mu(S_k) \\ &\le Cs \sum_{k=1}^l (\varepsilon + \sigma_{n,\alpha}(S_k)) \le C\|f\|_{1,\alpha}.\end{aligned}$$

Allowing ε to tend to zero we conclude with

$$\mu\{z \in U^n : B_\alpha[f] > s\} \leq C s^{-1} \|f\|_{1,\alpha}.$$

This is the desired inequality in (6). Thus the proof is complete.

(ii) Suppose that I_α is a compact operator from $A_\alpha^p(U^n)$ into $L^p(\mu)$. Let

$$f_\delta(z_1, z_2, \dots, z_n) = \prod_{i=1}^n \frac{\delta_i^{\beta-(\alpha+2)/p}}{(1-\overline{\eta_i} z_i)^\beta},$$

where $0 < \delta_j < 1$, $\beta > (\alpha+2)/p$, and $\eta_j = (1-\delta_j)e^{i(\theta_j^0+\delta_j/2)}$. These functions form a bounded subset of $A_\alpha^p(U^n)$, and tend to zero weakly as $\delta_i \to 0$. Since $|1-\overline{\eta_i} z_i| < 2\delta_i$, on regions $S(R)$,

$$\begin{aligned} |f_\delta(z_1, z_2, \dots, z_n)|^p &> \prod_{i=1}^n \frac{\delta_i^{(\beta-(\alpha+2)/p)^p}}{(2^\beta \delta_i^\beta)^p} \\ &= \prod_{i=1}^n \frac{1}{\delta_i^{\alpha+2} 2^{p\beta}}. \end{aligned}$$

Hence

$$\frac{\mu(S(R))}{\prod_{i-1}^n 2^{p\beta}\delta_i^{\alpha+2}} \leq \int_{\overline{U^n}} |f_\delta|^p \, d\mu =: \varepsilon(\delta),$$

where $\varepsilon(\delta) \to 0$ as $\delta_i \to 0$ for some i. Hence $\mu(S) \leq \varepsilon(\delta) 2^{n\beta p} \prod_{i=1}^n \delta_i^{\alpha+2}$; i.e.

$$\lim_{\delta_i \to 0} \sup_{\theta \in T^n} \frac{\mu(S(R))}{\prod_{i=1}^n \delta_i^{\alpha+2}} = 0.$$

Hence μ is a compact α-Carleson measure for $A_\alpha^p(U^n)$. Conversely, suppose that μ is a compact α-Carleson measure. We want to show that I_α is compact. By Lemma 1.4 it is sufficient to show that if $f_j \in A_\alpha^p(U^n)$ converges to zero weakly, as members of $L^p(\mu)$, then the sequence f_j converges to zero in norm. The hypothesis on μ states that given $\varepsilon > 0$ we can choose δ so small that

$$\sup_{\theta \in T^n} \frac{\mu(S(R))}{\prod_{i=1}^n \delta_i^{\alpha+2}} \leq \varepsilon(\delta) \tag{12}$$

for all $\delta_i < \delta$. Fix $\varepsilon > 0$ and choose δ so that (12) is satisfied. Write $\mu = \mu_1 + \mu_2$, where μ_1 is the restriction of μ to $(1-\delta)\overline{U^n}$ and μ_2 lives on the complement of this set in $\overline{U^n}$. Then, since $\mu_2 \leq \mu$,

$$\sup \frac{\mu_2(S(R))}{t^{n(\alpha+2)}} \leq \sup \frac{\mu(S(R))}{t^{n(\alpha+2)}},$$

where the supremums are extended over all $\theta \in T^n$ and for all $0 < t < \delta$. Since μ is a compact α-Carleson measure, by (12) the right-hand side of the above expression is less than ε, i.e. μ_2 is a compact Carleson measure (for A_α^p).

Proceeding exactly as in Theorem 2.3 we note that μ_2 satisfies the Carleson condition (for A^p_α) in part (i) with the constant C replaced by $C\varepsilon(\delta)$. Hence

$$\int_{\overline{U^n}} |f_j|^p \, d\mu_2 \le \sup \frac{\mu_2(S(R))}{t^{n(\alpha+2)}} \|f_j\|^p_{p,\alpha} \le C\varepsilon(\delta)\|f_j\|^p_{p,\alpha}.$$

Since f_j is a norm bounded family, the integral with respect to μ_2 can be made as small as desired. For δ fixed, the integral with respect to μ_1 can also be made arbitrarily small by choosing j sufficiently large. □

Theorem 2.5 can be generalized to a relation between the identity operator $I_\alpha : A^p_\alpha(U^n) \to L^q(\mu)$, $1 < p \le q < \infty$, and the measure μ. That is, it is possible to show that I_α is bounded if and only if $\mu(S(R)) \le C \prod_{i=1}^n \delta_i^{q(\alpha+2)/p}$, and likewise I_α is compact if and only if an analogous statement is true, with C replaced by ε. Since the Carleson condition is independent of p, and by part (iv) of Corollary 1.2 $A^2_{-1}(U^n) = H^2(U^n)$, we find that $I_\alpha : H^p \to L^p(\mu)$ is bounded if and only if $\mu(S(R)) \le C \prod_{i=1}^n \delta_i = Cm_n(R)$, and I_α is compact if and only if the above inequality holds with C replaced by ε. This is of course the same as Corollary 2.2. We conclude this section by giving two applications of Theorem 2.5. First, we shall use Theorem 2.5 to prove Proposition 1.5, and then we shall obtain a weighted Fejer-Riesz inequality.

Proof of Proposition 1.3. By Theorem 2.5 it is sufficient to prove the assertion for $p = 2$. By Corollary 1.2, $A^2_\alpha \subset A^2_\beta$ whenever $-1 < \alpha < \beta$. Suppose I_α is bounded on A^2_α. Then by Theorem 2.5(i), μ_α is an α-Carleson measure. Hence

$$\mu_\alpha(S(R)) \le C \prod_{i=1}^n \delta_i^{\alpha+2} \tag{1}$$

$S(R)$ here is the same as in definition (i) at the beginning of this section. Fix $S(R) = S$. On S, $1 - |z_j|^2 \le \delta_j$, so we have

$$\begin{aligned}\mu_\beta(S) &= \int_S \prod_{i=1}^n (1 - |z_i|^2)^\beta \, d\nu_n \\ &\le \prod_{i=1}^n \delta_i^{\beta-\alpha} \int_S \prod_{i=1}^n (1 - |z_i|^2)^\alpha \, d\nu_n \\ &= \prod_{i=1}^n \delta_i^{\beta-\alpha} \mu_\alpha(S).\end{aligned}$$

Hence

$$\mu_\beta(S) \le \prod_{i=1}^n \delta_i^{\beta-\alpha} \mu_\alpha(S). \tag{2}$$

Combining (2) and (1) we get

$$\mu_\beta(S) \le \prod_{i=1}^n \delta_i^{\beta-\alpha} \mu_\alpha(S) \le \prod_{i=1}^n \delta_i^{\beta-\alpha} C \prod_{i=1}^n \delta_i^{\alpha+2} = C \prod_{i=1}^n \delta_i^{\beta+2}.$$

Using Theorem 2.5 once again we conclude that I_β is a bounded operator on $A^2_\beta(U^n)$.

To show compactness we replace the constant C by $\varepsilon(\delta)$ throughout the above argument. □

Corollary 2.6. *If* $f \in A^p_\alpha(U^n)$ *and* $\alpha > 0$, *then*

$$\int_{[0,1]^n} |f|^p \prod_{i=1}^n (1-r_i^2)^{\alpha+1}\, dr_1 \cdots dr_n \le C \int_{\overline{U^n}} |f|^p \prod_{i=1}^n (1-|z_i|^2)^\alpha\, d\sigma_n(z).$$

Proof. Let $\mu = \prod_{i=1}^n (1-r_i^2)^{\alpha+1}\, dr_1 \cdots dr_n$. Then

$$\mu(S(R)) \le C \prod_{i=1}^n \delta_i^{2+\alpha}.$$

Hence μ is an α-Carleson measure on $A^p_\alpha(U^n)$ and thus, by Theorem 2.5,

$$\int_{\overline{U^n}} |f|^p\, d\mu \le C \int_{\overline{U^n}} |f|^p \prod_{i=1}^n (1-|z_i|^2)^\alpha\, d\sigma_n(z),$$

i.e.

$$\int_{[0,1]^n} |f|^p \prod_{i=1}^n (1-r_i^2)^{\alpha}\, dr_1 \cdots dr_n \le C \int_{\overline{U_n}} |f|^p \prod_{i=1}^n (1-|z_i|^2)^\alpha\, d\sigma_n(z). \quad \square$$

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