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THE STABILITY OF CERTAIN FUNCTIONAL EQUATIONS

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ABSTRACT. The aim of this paper is to prove the stability (in the sense of Ulam) of the functional equation:

$$f(t) = \alpha(t) + \beta(t) f(\phi(t)),$$

where α and β are given complex valued functions defined on a nonempty set S such that $\sup\{|\beta(t)| : t \in S\} < 1$ and ϕ is a given mapping of S into itself.

INTRODUCTION

Questions concerning the stability of functional equations seem to have originated with S. M. Ulam in the 1940s (see [2, 3] and the survey paper of Hyers [4]). One of the first assertions to be proved in this direction is the following result, essentially due to Hyers, that answered a question of Ulam.

Theorem. *Suppose S is an additive semigroup, E is a Banach space, $f : S \to E$, $\delta > 0$, and*

$$\|f(x+y) - f(x) - f(y)\| \le \delta \quad \textit{for all } x, y \in S. \tag{1}$$

Then there is a unique function $a : S \to E$ such that

$$a(x+y) = a(x) + a(y) \quad \textit{for all } x, y \in S, \tag{2}$$

and

$$\|f(x) - a(x)\| \le \delta \quad \textit{for } x \in S. \tag{3}$$

This assertion is usually summarized by saying that the Cauchy functional equation (2) is stable (in the sense of Ulam).

The proof proceeds as follows. First note that

$$\|f(2x)/2 - f(x)\| \le \tfrac{\delta}{2} \quad \text{for all } x \in S. \tag{4}$$

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Then one proves, by induction, that $\{f(2^n x)/2^n\}_{n=1}^{\infty}$ is a Cauchy sequence in E. Denoting its limit by $a(x)$, one then proves that (2) and (3) hold. The same idea can be used to prove the stability of the equation:

$$f(2x)/2 = f(x). \tag{5}$$

In fact the same technique can be used to prove the

Proposition. *Suppose S is a nonempty set, $\phi : S \to S$, E is a Banach space, λ is a scalar, $|\lambda| < 1$, $\delta > 0$, and $f : S \to E$ such that*

$$\|\lambda f(\phi(x)) - f(x)\| \le \delta \quad \textit{for all } x \in S. \tag{6}$$

Then there exists a unique $h : S \to E$ such that

$$\lambda h(\phi(x)) = h(x) \quad \textit{for all } x \in S, \tag{7}$$

and

$$\|f(x) - h(x)\| \le \delta/(1 - |\lambda|) \quad \textit{for all } x \in S. \tag{8}$$

Proof. Let ϕ^k denote the kth iterate of ϕ for $k = 0, 1, 2, \dots$. Thus $\phi^0(x) = x$, $\phi^1(x) = \phi(x)$, $\phi^2(x) = \phi(\phi(x))$, etc. for all $x \in S$.

By induction, it follows from (6) that

$$\|\lambda^{k+1} f(\phi^{k+1}(x)) - \lambda^k f(\phi^k(x))\| \le |\lambda|^k \delta \quad \text{for all } x \in S \text{ and } k = 0, 1, 2, \dots.$$

Hence, if m and n are nonnegative integers and $m < n$, then

$$\begin{aligned} \|\lambda^n f(\phi^n(x)) - \lambda^m f(\phi^m(x))\| &\le (|\lambda|^{n-1} + \cdots + |\lambda|^m)\delta \\ &\le \frac{|\lambda|^m \delta}{1 - |\lambda|} \quad \text{for all } x \in S. \end{aligned} \tag{9}$$

Thus $\{\lambda^n f(\phi^n(x))\}_{n=1}^{\infty}$ is a Cauchy sequence in E for each $x \in S$. Let $h(x) = \lim_{n\to\infty} \lambda^n f(\phi^n(x))$ for $x \in S$. Then (7) clearly holds.

If we choose $m = 0$ in (9) we find that

$$\|\lambda^n f(\phi^n(x)) - f(x)\| \le \delta/(1 - |\lambda|) \quad \text{for all } x \in S,$$

and all $n = 1, 2, \dots$, and this implies (8). □

Most known stability theorems for functional equations concern functional equations in "several variables" in the sense of Aczél [1]. Equation (2) is the best-loved representative of this class. Equation (7), whose stability we have just proved, is often referred to as a functional equation in a "single variable." Such equations are the subject of the book by Kuczma [5]. The aim of this paper is to generalize the above proposition by proving a stability result for the general functional equation:

$$f(t) = F(t, f(\phi(t))).$$

We refer the reader to [5] for numerous results and references concerning this equation and particular cases thereof.

Main results

Our analysis is based on the following variant of Banach's fixed point theorem that is likely included among the thousands of published generalizations thereof.

Theorem 1. *Suppose* (Y, ρ) *is a complete metric space and* $T : Y \to Y$ *is a contraction (for some* $\lambda \in [0, 1)$, $\rho(T(x), T(y)) \leq \lambda\rho(x, y)$ *for all* $x, y \in Y$*). Also suppose that* $u \in Y$, $\delta > 0$, *and*

$$\rho(u, T(u)) \leq \delta.$$

Then there exists a unique $p \in Y$ *such that* $p = T(p)$. *Moreover,* $\rho(u, p) \leq \delta/(1-\lambda)$.

Proof. Define a sequence $\{y_n\}_{n=0}^{\infty}$ in Y by decreeing that $y_0 = u$ and $y_{n+1} = T(y_n)$ for all $n = 0, 1, 2, \ldots$. According to the well-known proof of Banach's fixed point theorem, $\{y_n\}_{n=1}^{\infty}$ converges, say to p, p is the unique fixed point of T, and

$$\rho(y_n, p) \leq \frac{\lambda^n}{1-\lambda}\rho(y_1, y_0) \quad \text{for } n = 0, 1, 2, \ldots.$$

Thus

$$\rho(u, p) \leq \rho(u, T(u)) + \rho(T(u), p) \leq \delta + \rho(T(u), T(p)) \leq \delta + \lambda\rho(u, p)$$

so that $\rho(u, p) \leq \delta/(1-\lambda)$. □

Theorem 2. *Suppose* S *is a nonempty set;* (X, d) *is a complete metric space;* $\phi : S \to S$, $F : S \times X \to X$, $0 \leq \lambda < 1$; *and*

$$d(F(t, u), F(t, v)) \leq \lambda d(u, v) \quad \textit{for all } t \in S \textit{ and all } u, v \in S.$$

Also suppose that $g : S \to X$, $\delta > 0$, *and*

$$d(g(t), F(t, g(\phi(t)))) \leq \delta \quad \textit{for all } t \in S. \tag{10}$$

Then there is a unique function $f : S \to X$ *such that*

$$f(t) = F(t, f(\phi(t))) \quad \textit{for all } t \in S, \tag{11}$$

and

$$d(f(t), g(t)) \leq \delta/(1-\lambda) \quad \textit{for all } t \in S. \tag{12}$$

Proof. Let $Y = \{a : S \to X \mid \sup\{d(a(t), g(t)) \mid t \in S\} < +\infty\}$. For $a, b \in Y$ define $\rho(a, b) = \sup\{d(a(t), b(t)) \mid t \in S\}$. Then $g \in Y$, ρ is a metric on Y, and convergence with respect to ρ means uniform convergence on S with respect to d. Moreover, the completeness of X with respect to d implies the completeness of Y with respect to ρ.

For $a \in Y$ define $T(a) : S \to X$ by

$$(T(a))(t) = F(t, a(\phi(t))) \quad \text{for } t \in S.$$

Then T maps Y into Y. If $a, b \in Y$ then for all $t \in S$,

$$\begin{aligned} d((T(a))(t), (T(b))(t)) &= d(F(t, a(\phi(t))), F(t, b(\phi(t)))) \\ &\leq \lambda d(a(\phi(t)), b(\phi(t))) \\ &\leq \lambda\rho(a \circ \phi, b \circ \phi) \\ &\leq \lambda\rho(a, b). \end{aligned}$$

Thus,

$$\rho(T(a), T(b)) \leq \lambda \rho(a, b) \quad \text{for all } a, b \in Y.$$

But (10) means that $\rho(g, T(g)) \leq \delta$. Hence, according to Theorem 1, there is a unique f in Y such that $f = T(f)$ and $\rho(g, f) \leq \delta/(1-\lambda)$. That is, (11) and (12) hold. □

We will now use Theorem 2 to deduce stability assertions concerning the functional equation:

$$f(t) = \alpha(t) + \beta(t) f(\phi(t)).$$

Theorem 3. *Suppose S is a nonempty set; E is a real (or complex) Banach space; $\phi : S \to S$, $\alpha : S \to E$, $\beta : S \to \mathbb{R}$ (or $\mathbb{C}$), $0 \leq \lambda < 1$; and $|\beta(t)| \leq \lambda$ for all $t \in S$. Also suppose that $g : S \to E$, $\delta > 0$, and*

$$\|g(t) - \{\alpha(t) + \beta(t) g(\phi(t))\}\| \leq \delta \quad \textit{for all } t \in S. \tag{13}$$

Then there exists a unique function $f : S \to E$ such that

$$f(t) = \alpha(t) + \beta(t) f(\phi(t)) \tag{14}$$

and

$$\|f(t) - g(t)\| \leq \delta/(1-\lambda) \quad \textit{for all } t \in S. \tag{15}$$

Proof. The result follows from Theorem 2 by letting

$$F(t, x) = \alpha(t) + \beta(t)x \quad \text{for } (t, x) \in S \times E. \quad \square$$

Similarly we can prove

Theorem 4. *Suppose S is a nonempty set; A is a Banach algebra; $\phi : S \to S$, $\alpha : S \to A$, $\beta : S \to A$; $0 \leq \lambda < 1$, $\|\beta(t)\| \leq \lambda$ for all $t \in S$, $g : S \to A$, $\delta > 0$; and* (13) *holds. Then there is a unique function $f : S \to A$ satisfying* (14) *and* (15).

Acknowledgment

A referee has noted that our proposition is a particular case of Theorem 1 and Corollary 3 of G. L. Forti, *An existence and stability theorem for a class of functional equations*, Stochastica **4** (1980), 23–30.

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