PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 3, July 1991

AN $n \times n$ MATRIX OF LINEAR MAPS OF A C^*-ALGEBRA

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(Communicated by Paul S. Muhly)

ABSTRACT. Every positive $n \times n$ matrix of linear functionals on a C^*-algebra is completely positive. [3, Theorem 2.1] can be extended to the case of a bounded $n \times n$ matrix of linear functionals.

1. INTRODUCTION

Let M_n denote the C^*-algebra of complex $n \times n$ matrices generated as a linear space by the matrix units E_{ij} $(i, j = 1, 2, \ldots, n)$ and let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H. Let A and B be C^*-algebras and let $L: A \to B$ be a bounded linear map, the map L is called positive provided that $L(a)$ is positive whenever a is positive. The map L is called completely positive if $L \otimes I_n: A \otimes M_n \to B \otimes M_n$ defined by $L \otimes I_n(a \otimes b) = L(a) \otimes b$ is positive for all n. L is completely bounded if $\sup_n \|L \otimes I_n\|$ is finite, and we let $\|L\|_{\mathrm{cb}} = \sup_n \|L \otimes I_n\|$. We define $L^*(a) = L(a)^*$. Given $S \subseteq B(H)$, we let S' denote its commutant. An $n \times n$ matrix (f_{ij}) of linear functionals on a C^*-algebra A is positive (or an n-positive linear functional on A [3, p. 1]) if $(f_{ij}(a_{ij}))$ is positive whenever (a_{ij}) is a positive element in $A \otimes M_n$. The paper [3] does not show that an n-positive linear functional on a C^*-algebra is completely positive. In this paper we prove the fact, generalize [3, Theorem 2.1], and develop an $n \times n$ matrix of linear maps from a C^*-algebra to $B(H)$.

2. A POSITIVE $n \times n$ MATRIX OF LINEAR FUNCTIONALS

We will apply the following well-known theorem [7, Corollary 2.3] later in this paper.

Theorem 2.1. *Let F be a linear map from a C^*-algebra A to M_n and let the functional $f: A \otimes M_n \to \mathbb{C}$ be defined by $f(a \otimes E_{ij}) = (F(a))_{ij}$. If f is positive, then F is completely positive.*

The following theorem shows that positivity implies complete positivity.

Received by the editors October 23, 1989 and, in revised form, August 23, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L05.
Key words and phrases. Positive map, completely positive linear map, completely bounded linear map, commutant representation.

Theorem 2.2. *Let* $F = (f_{ij}): A \otimes M_n \to M_n(\mathbb{C})$ *be a positive* $n \times n$ *matrix of linear functionals on* A, *then* F *is completely positive.*

Proof. Define the map $L: M_n \otimes (M_n(A)) \to C$ by

$$L((a_{kl}) \otimes E_{ij}) = (F((a_{kl})))_{ij} = f_{ij}(a_{ij}).$$

By [10, p. 193] we know that every positive element of $M_n \otimes (M_n(A))$ is a sum of positive elements of the form $(a_i^* a_j)$. Let $(a_i^* a_j)$ be an $n^2 \times n^2$ positive matrix in $M_n \otimes (M_n(A))$, then

$$\begin{aligned}
L((a_i^* a_j)) &= L\left(\begin{pmatrix} a_1 & \cdots & a_{n^2} \\ & 0 & \end{pmatrix}^* \begin{pmatrix} a_1 & \cdots & a_{n^2} \\ & 0 & \end{pmatrix}\right) \\
&= L\left(\left(\begin{pmatrix} a_1^* & \\ \vdots & 0 \\ a_n^* & \end{pmatrix} \otimes E_{11} + \begin{pmatrix} a_{n+1}^* & \\ \vdots & 0 \\ a_{n^2}^* & \end{pmatrix} \otimes E_{21}\right)\right. \\
&\qquad \left. \times \left(\begin{pmatrix} a_1 & \cdots & a_n \\ & 0 & \end{pmatrix} \otimes E_{11} + \begin{pmatrix} a_{n+1} & \cdots & a_{n^2} \\ & 0 & \end{pmatrix} \otimes E_{12}\right)\right) \\
&= L\left(\begin{pmatrix} a_1^* a_1 & \cdots & a_1^* a_n \\ \vdots & & \vdots \\ a_n^* a_1 & \cdots & a_n^* a_n \end{pmatrix} \otimes E_{11} + \begin{pmatrix} a_1^* a_{n+1} & \cdots & a_1^* a_{n^2} \\ \vdots & & \vdots \\ a_n^* a_{n+1} & \cdots & a_n^* a_{n^2} \end{pmatrix} \otimes E_{12}\right. \\
&\qquad \left. + \begin{pmatrix} a_{n+1}^* a_1 & \cdots & a_{n+1}^* a_n \\ \vdots & & \vdots \\ a_{n^2}^* a_1 & \cdots & a_{n^2}^* a_n \end{pmatrix} \otimes E_{21} + \begin{pmatrix} a_{n+1}^* a_{n+1} & \cdots & a_{n+1}^* a_{n^2} \\ \vdots & & \vdots \\ a_{n^2}^* a_{n+1} & \cdots & a^* +_{n^2} a_{n^2} \end{pmatrix} \otimes E_{22}\right) \\
&= f_{11}(a_1^* a_1) + f_{12}(a_1^* a_{n+2}) + f_{21}(a_{n+2}^* a_1) + f_{22}(a_{n+2}^* a_{n+2}) \\
&= F\left(\begin{pmatrix} a_1 & a_{n+2} & 0 & \cdots & 0 \\ & 0 & & & \end{pmatrix}^* \begin{pmatrix} a_1 & a_{n+2} & 0 & \cdots & 0 \\ & 0 & & & \end{pmatrix}\right) \geq 0.
\end{aligned}$$

Thus L is positive. By Theorem 2.1 F is completely positive.

The following theorem generalizes [3, Theorem 2.1].

Theorem 2.3. *Let* $F = (f_{ij}): A \otimes M_n \to M_n(\mathbb{C})$ *be an* $n \times n$ *matrix of linear functions on* A *as defined by* $F((a_{ij})) = (f_{ij}(a_{ij}))$. *If* F *is bounded, then there is a representation* π *of* A *on a Hilbert space* K, *and* $2n$ *vectors* $x_1, \ldots, x_n, y_1, \ldots, y_n$ *in* K *such that* $\{x_1, \ldots, y_n\}$ *is a generating set for* $\pi(A)$ *on* K,

$$f_{ij}(a) = \langle \pi(a) y_j, x_i \rangle \quad \textit{and} \quad \|F\|_{\text{cb}} = \max_i \{\|x_i\|^2, \|y_i\|^2\}.$$

Proof. From [6] F is completely bounded. Let D be the C^*-algebra of diagonal matrices in $M_n(\mathbb{C})$, then F is D-bihomomorphism (or D-bimodule) [9, Definition 2.1]. By [9, Theorem 2.5] there exist D-bimodule completely positive maps $\phi_i: A \otimes M_n \to M_n(\mathbb{C})$, with $\|\phi_i\|_{\text{cb}} = \|F\|_{\text{cb}}$ $(i = 1, 2)$ such that the map

$$\begin{pmatrix} \phi_1 & F \\ F^* & \phi_2 \end{pmatrix} : A \otimes M_{2n} \to M_{2n}(\mathbb{C})$$

is completely positive. Hence it is a positive $2n \times 2n$ matrix of linear functions on A. By [3, Theorem 2.1], there exist $2n$ vectors $x_1, \dots, y_n$ in K such that $f_{ij}(a) = \langle \pi(a)y_j, x_i \rangle$, $f^*_{ij}(a) = \langle \pi(a)x_i, y_j \rangle$ $(i, j = 1, \dots, n)$ and $\|F\|_{\mathrm{cb}} = \max_i\{\|x_i\|^2\} = \max_i\{\|y_i\|^2\}$.

In Theorem 2.3, (f_{ij}) has the representation:

Corollary 2.4. *Let V and W be operators from $\mathbb{C}^n$ to $\overbrace{K \oplus \cdots \oplus K}^{n}$ defined by $Ve_i = (0, 0, \dots, x_i, \dots, 0)$ and $We_i = (0, 0, \dots, y_i, \dots, 0)$, then $(f_{ij})((a_{ij})) = V^* \pi \otimes I_n((a_{ij}))W$.*

Corollary 2.5. $\|(f_{ij})\|_{\mathrm{cb}} \le 1$ *if and only if there is a representation π of A on a Hilbert space K, and $2n$ vectors $x_1, \dots, x_n, y_1, \dots, y_n$ in K such that*

$$f_{ij}(a) = \langle \pi(a)y_j, x_i \rangle \quad \textit{with } \|x_i\| \le 1 \textit{ and } \|y_i\| \le 1 \ (i, j = 1, \dots, n).$$

Proof. "$\Rightarrow$" By Theorem 2.3, we have $f_{ij}(a) = \langle \pi(a)y_j, x_i \rangle$ and $\max_i\{\|x_i\|^2\} = \max_i\{\|y_i\|^2\} = \|(f_{ij})\|_{\mathrm{cb}} \le 1$. Hence $\|x_i\| \le 1$ and $\|y_i\| \le 1$ $(i = 1, \dots, n)$.

"$\Leftarrow$" We define the map

$$V: \mathbb{C}^{2n} \to \overbrace{K \oplus \cdots \oplus K}^{2n}$$

$$\text{by } Ve_i = (0, \dots, \underset{i\text{th slot}}{x_i}, \dots, 0) \text{ and } Ve_{n+i} = (0, \dots, \underset{(n+i)\text{th slot}}{y_i}, \dots, 0)$$

for $i = 1, \dots, n$. Then the map

$$\begin{pmatrix} (\langle \pi(a)x_j, x_i \rangle) & (\langle \pi(a)y_j, x_i \rangle) \\ (\langle \pi(a)x_i, y_j \rangle) & (\langle \pi(a)y_j, y_i \rangle) \end{pmatrix}_{2n \times 2n} = V^* \pi \otimes I_{2n}(a)V$$

is completely positive. Hence the completely bounded norm of the map is $\max\{\|x_i\|^2, \|y_i\|^2\} \le 1$. By [2], we have $\|(f_{ij})\|_{\mathrm{cb}} \le 1$.

From Corollary 2.5, we obtain the property of Schur Product [4, pp. 110–112]:

Corollary 2.6. *Let $A = (a_{ij})$ and $S_A: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be defined by $S_A((b_{ij})) = (a_{ij}b_{ij})$. $\|S_A\|_{\mathrm{cb}} \le 1$ iff there exist $2n$ vectors $x_1, \dots, x_n, y_1, \dots, y_n$ such that $A = (\langle y_j, x_i \rangle)$ with $\|x_i\| \le 1$, $\|y_i\| \le 1$ and $\|S_A\|_{\mathrm{cb}} = \max_i\{\|x_i\|^2, \|y_i\|^2\}$.*

Proof. Since S_A is D-bimodule, we have the corollary from Corollary 2.5.

In general, we consider an $n \times n$ matrix of linear maps from a C^*-algebra to $B(H)$.

Proposition 2.7. *If the map $(f_{ij}): A \otimes M_n \to B(H) \otimes M_n$ defined by $(f_{ij})((a_{ij})) = (f_{ij}(a_{ij}))$ is completely positive, then there is a representation π of A on a*

Hilbert space K, an isometry $V: H \to K$ and an operator $T_{ij} \in \pi(A)'$ such that $[\pi(A)VH]$ is dense in K and $f_{ij}(\cdot) = V^* T_{ij}\pi(\cdot)V$ with $(T_{ij}) \geq 0$.

Proof. It is not difficult to see that the map

$$\begin{pmatrix} f_{ii} & f_{ij} \\ f_{ji} & f_{jj} \end{pmatrix} : A \otimes M_2 \to B(H) \otimes M_2$$

is completely positive. By [7, Lemma 2.3], we have that f_{ii} is completely positive, $f_{ij}^* = f_{ji}$ and $(f_{ii} + f_{jj})/2 - (\pm \operatorname{Re} z f_{ij})$ is completely positive for $(i, j = 1, 2, \dots, n$, and $|z| = 1)$. Hence $\frac{1}{2}(\sum_{i=1}^n f_{ii}) - (\pm \operatorname{Re} z f_{ij})$ is completely positive for $i, j = 1, 2, \dots, n$.

By [5, Theorem 2.10], there is a representation π of A on a Hilbert space K, an isometry $V: H \to K$ and an operator $T_{ij} \in \pi(A)'$ such that $[\pi(A)VH]$ is dense in K and

$$f_{ij}(\cdot) = V^* T_{ij}\pi(\cdot)V.$$

By [5, Proposition 2.6], we have that $(T_{ij}) \geq 0$.

Acknowledgment

The author gratefully acknowledges several valuable conversations with Professor Vern I. Paulsen.

References

1. W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1963), 141–224.
2. U. Haagerup, *Injectivity and decomposition of completely bounded maps*, Operator Algebras and Their Connections with Topology and Ergodic Theory (Busteni, 1983), Lecture Notes in Math., vol. 1132, Springer-Verlag, Berlin and New York, 1985, pp. 70–122.
3. A. Kaplan, *Multi-states on C^*-algebras*, Proc. Amer. Math. Soc. **106** (1989), 437–446.
4. V. I. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Math., vol. 146, Longman, London, 1986.
5. V. I. Paulsen and C.-Y. Suen, *Commutant representations of completely bounded maps*, J. Operator Theory **13** (1985), 87–101.
6. R. R. Smith, *Completely bounded maps between C^*-algebras*, J. London Math. Soc. **27** (1983), 157–166.
7. R. R. Smith and J. D. Ward, *Matrix ranges for Hilbert space operators*, Amer. J. Math. **102** (1980), 1041–1081.
8. W. F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.
9. C.-Y. Suen, *Completely bounded maps on C^*-algebras*, Proc. Amer. Math. Soc. **93** (1985), 81–87.
10. M. Takasaki, *Theory of operator algebra* 1, Springer-Verlag, Berlin, 1979.

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