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SIMPLE C^*-ALGEBRAS WITH CONTINUOUS SCALES AND SIMPLE CORONA ALGEBRAS

HUAXIN LIN

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ABSTRACT. It is shown that the corona algebra $M(A)/A$ of a separable simple C^*-algebra A is simple if and only if A has a continuous scale or A is elementary. It is also shown that simple C^*-algebras with continuous scales are algebraically simple.

1. INTRODUCTION AND PRELIMINARIES

Let A be a C^*-algebra and A^{**} the enveloping von Neumann algebra of A. The multiplier algebra $M(A)$ is the idealiser of A in A^{**}. We denote by K the C^*-algebra of all compact operators on an infinite-dimensional separable Hilbert space H and by $B(H)$ the C^*-algebra of all bounded operators on H. It is well known that $M(K) = B(H)$ and $M(K)/K$ is simple. It is natural to ask when the corona algebra $M(A)/A$ is simple. Zhang [17] and Rørdam [16] showed that if A is stable then the simplicity of the corona algebra of A implies the simplicity of A. The ideal structure of $M(A)/A$ for a simple C^*-algebra has been studied in [8, 9, 11–13]. In this paper we shall show that if A is a separable simple C^*-algebra then the corona algebra is simple if and only if either A has a continuous scale or A is elementary. (We say A is elementary if $A \cong K$ or $A \cong M_n$, the $n \times n$ matrices over $\mathbb{C}$.) We also show that if A has a continuous scale then A is algebraically simple. In other words, the simplicity of corona algebra of A implies the algebraic simplicity of A. (But the converse is not true.)

Let A be a C^*-algebra, a and b be in A. Following [5–7], we write $a \lessapprox b$ if there are x, y in $\widetilde{A}$, the C^*-algebra obtained from A by adjoining a unit, such that $a = xby$. We write $a \lesssim b$ if there is a sequence $\{x_n\}$ in A such that $x_n \lessapprox b$ and $x_n \to a$ in norm. We write $a \approx b$ if $a \lessapprox b$ and $b \lessapprox a$, and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. One has $x \sim x^*x \sim x^*$. If a and b are positive, then $a \lesssim b$ if and only if there is a sequence $\{r_k\}$ in A such that $r_k^* b r_k \to a$.

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Moreover, if $0 \le a \le b$, then $a \lesssim b$. For the details of the relations "$\lesssim\!\!\!\!\approx$" and "$\lesssim$," readers are referred to [5–7].

Further, we define a relation $\tilde{<}$ on A_+ by $a \tilde{<} b$ if there is $x \in A$ such that $a = x^*x$ and $xx^* \in \mathrm{Her}(b)$, where $\mathrm{Her}(b)$ is the hereditary C^*-subalgebra generated by b. If $a, b \in A_+$, $a \tilde{<} b$, then $a \lesssim b$. On the other hand if $a \lessapprox b$, then $a \tilde{<} b$ by the proof of [5, Lemma 1.7].

For each $\varepsilon > 0$ define a continuous function $f_\varepsilon \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_\varepsilon(t) = \begin{cases} 0, & t \le \varepsilon; \\ \varepsilon^{-1}(t-\varepsilon), & \varepsilon \le t \le 2\varepsilon; \\ 1, & t \ge 2\varepsilon. \end{cases}$$

When a is a positive element in A, we shall denote by $[a]$ the range projection of a in the enveloping von Neumann algebra A^{**}. Suppose that A is σ-unital (and nonunital), and let e be a strictly positive element of A. By choosing a proper sequence of continuous functions h_n, we can construct an approximate identity $\{e_n\}$ $(e_n = h_n(e))$ for A satisfying:

(i) $g_n = e_n - e_{n-1}$ $(e_0 = 0)$, $g_m g_n = 0$ if $|m-n| \ge 2$, and $\|g_n\| = 1$.
(ii) There are $a_n \in A_+$, $\|a_n\| = 1$ such that $0 \le a_n \le [a_n] \le g_n$, $a_n g_n = g_n a_n = a_n$ and $a_n g_m = g_m a_n = 0$ if $n \ne m$.

Any subsequence $\{e_{n_k}\}$ of $\{e_n\}$ is also an approximate identity satisfying (i) and (ii). We assume that any approximate identity appearing in this paper satisfies conditions (i) and (ii).

We shall denote by $P(A)$ the Pedersen ideal of A. We say that x is orthogonal to y and write $x \perp y$ if $xy = yx = x^*y = yx^* = 0$.

2. Simple corona algebras

Lemma 2.1. *Let A be a σ-unital, nonunital, nonelementary simple C^*-algebra, and $\{e_n\}$ be an approximate identity. Set*

$$\begin{aligned} I_0 = \{x \in M(A) : \forall a \in A_+,\ \|a\| \ne 0,\ \exists n_0, \\ \ni (e_m - e_n)x^*x(e_m - e_n) \lesssim a,\ \forall m > n \ge n_0\}. \end{aligned}$$

Then I_0 is a (*-invariant) ideal of $M(A)$.

Proof. Clearly I_0 is a *-invariant subset of $M(A)$, and if $\lambda \in \mathbb{C}$, then $\lambda I_0 \subset I_0$. Suppose that $a \in I_0$, $b \in M(A)$; then

$$\begin{aligned} (e_m - e_n)a^*b^*ba(e_m - e_n) &\sim ba(e_m - e_n)^2 a^* b^* \\ &\lesssim a(e_m - e_n)^2 a^* \\ &\sim (e_m - e_n)a^*a(e_m - e_n) \end{aligned}$$

for any $m > n$. Therefore $ba \in I_0$. Since I_0 is *-invariant, $ab \in I_0$. Now suppose that $a, b \in I_0$; then

$$\begin{aligned}(e_m - e_n)(a+b)^*(a+b)(e_m - e_n) \\ &= (e_m - e_n)a^*a(e_m - e_n) + (e_m - e_n)b^*b(e_m - e_n) \\ &\quad + (e_m - e_n)a^*b(e_m - e_n) + (e_m - e_n)b^*a(e_m - e_n)\end{aligned}$$

for all $m > n$. Let $x \in A_+$ with $\|x\| = 1$. Since A is nonelementary, $\mathrm{Her}(f_{1/2}(x))$ is also nonelementary. We can find $x_i, y_i \in \mathrm{Her}(f_{1/2}(x))$ such that $x_iy_i = y_ix_i = x_i$, and $y_i \perp y_j$ if $i \neq j$, $x_i, y_i \geq 0$, $i = 1, 2, 3, 4$. Since a, b, a^*b and $b^*a \in I_0$, there is n_0 such that

$$\begin{aligned}(e_m - e_n)a^*a(e_m - e_n) &\precsim f_\varepsilon(x_1) \leq y_1, \\ (e_m - e_n)b^*b(e_m - e_n) &\precsim f_\varepsilon(x_2) \leq y_2, \\ (e_m - e_n)a^*b(e_m - e_n) &\precsim (e_m - e_n)(a^*b)^*(a^*b)(e_m - e_n) \\ &\precsim f_\varepsilon(x_3) \leq y_3, \\ (e_m - e_n)b^*a(e_m - e_n) &\precsim (e_m - e_n)(b^*a)^*(b^*a)(e_m - e_n) \\ &\precsim f_\varepsilon(x_4) \leq y_4,\end{aligned}$$

if $m > n \geq n_0$.

Since $y_i \perp y_j$ if $i \neq j$, by [6, Proposition 1.1],

$$(e_m - e_n)(a+b)^*(a+b)(e_m - e_n) \precsim \sum_{i=1}^{4} y_i \leq f_{1/4}(x) \precsim x$$

if $\frac{1}{4} > \varepsilon > 0$, $m > n > n_0$.

This completes the proof.

2.2. We denote by I the closure of I_0. Clearly I is a closed ideal of $M(A)$ containing A. From Remark 2.9, we see that I does not depend on the choices of $\{e_n\}$ if A is separable.

Lemma 2.3. *Let A be a simple C^*-algebra, a a positive element in A with $\|a\| = 1$, $z_i \in P(A)_+$, $i = 1, 2, \ldots, n$, and $0 < \delta \leq \frac{1}{4}$. Then there is $b \in A_+$ such that $0 \leq b \leq f_\delta(a)$, $\|b\| = 1$ and $b \precsim z_i$, $i = 1, 2, \ldots, n$.*

Proof. Suppose that $1 > \sigma \geq \frac{1}{2}$. Since $f_\sigma(a)$, $z_1 \in P(A)$, there are $r_j, s_j \in A$, $j = 1, 2, \ldots, m$, such that

$$f_\sigma(a) = \sum_{j=1}^{m} r_j z_1 s_j \leq \frac{1}{2}\left(\sum_{j=1}^{m} r_j z_1 r_j^* + \sum_{j=1}^{m} s_j^* z_1 s_j\right).$$

By asymmetric Riesz decomposition (see [15, 1.4.10] for example), there are $b_j \in A$, $j = 1, 2, \ldots, 2m$, such that $f_\sigma(a) = \sum_{j=1}^{2m} b_j^* b_j$, $b_j b_j^* \leq \frac{1}{2}(z_1^{1/2} r_j^* r_j z_1^{1/2})$ if $1 \leq j \leq m$, and $b_j b_j^* \leq \frac{1}{2}(z_1^{1/2} s_j s_j^* z_1^{1/2})$, if $m < j \leq 2m$.

We may assume that $b_1 \neq 0$ and take $a_1 = b_1^* b_1 / \|b_1^* b_1\|$; then $a_1 \precsim z_1$ and $a_1 \leq f_\delta(a)$.

Repeating the above argument, by induction, we see that Lemma 2.3 holds.

Lemma 2.4. *If A is a separable, nonunital, and nonelementary simple C^*-algebra, let I be the ideal defined in 2.2. Then I contains A properly.*

Proof. Suppose that $\{x_n\}$ is a dense subset of the unit ball of A. Put $z_n = (x_n^* x_n)^{1/2}$, $B_n = \mathrm{Her}(f_{1/2}(z_n))$. Since B_n is antiliminal, by [1, p. 67], there is $y_n \in B_n$ such that $y_n \geq 0$ and $\mathrm{sp}(y_n) = [0, 1]$. Therefore, there are $\{z_{n,i}\}_{i=1}^{\infty}$, $\{d_{n,i}\}_{i=1}^{\infty}$ in the C^*-subalgebra $C^*(y_n)$ generated by y_n such that $d_{n,i} \perp d_{n,j}$, $i \neq j$, $d_{n,i} z_{n,i} = z_{n,i} d_{n,i} = z_{n,i}$, $0 \leq z_{n,i} \leq d_{n,i} \leq 1$, and $\|z_{n,i}\| = 1$, $i = 1, 2, \dots$.

Notice $z_{n,i}, d_{n,i} \in B_n \subset P(A)$ and take $a_n \leq g_n$ as in 1 with $\|a_n\| = 1$. For each n, by Lemma 2.3, there is $0 \leq b_n \leq f_{1/4}(a_n)$ $(\leq [a_n] \leq g_n)$ such that $b_n \tilde{<} z_{i,j}$, $j = 1, 2, \dots, n+1-i$, $i = 1, 2, \dots, n$, and $\|b_n\| = 1$. Put $b = \sum_{n=1}^{\infty} b_n$. Since $0 \leq b_n \leq f_{1/4}(a_n) \leq g_n$, $b \in M(A)_+$. Let $x \in A_+$ with $\|x\| = 1$ and $0 < \varepsilon < \frac{1}{4}$. There is N such that $\|x - x_N\|$ small enough that

$$\|f_\varepsilon(x) - f_\varepsilon(z_N)\| < \varepsilon/4$$

(see [13, 2.2]). Then, by [16, 2.2], for some $r \in A$,

$$f_{1/4}(z_N) \leq f_{\varepsilon/2}(f_\varepsilon(z_N)) \leq r f_\varepsilon(x) r^* \lessapprox f_\varepsilon(x).$$

For any $m > N$, since $b_i \tilde{<} z_{N,i}$, $N \leq i \leq m$ and $z_{N,i} \perp z_{N,j}$ if $i \neq j$,

$$\sum_{i=N}^{m} b_i \lesssim \sum_{i=N}^{m} z_{N,i} \leq f_{1/4}(z_N) \lesssim f_\varepsilon(x) \lesssim x.$$

Therefore, $b = \sum_{i=1}^{\infty} b_i \in I$. However, since $\|b_i\| = 1$, $b \notin A$.

Definition 2.5. Let A be a σ-unital, nonunital, and nonelementary simple C^*-algebra and $\{e_n\}$ be an approximate identity for A. We say that A has a continuous scale if for any $x \in A_+$ with $\|x\| \neq 0$, there is n_0 for any $m > n \geq n_0$,

$$(e_m - e_n) \lesssim x.$$

We also use the convention that unital simple C^*-algebras have continuous scales. It should be noted that the property of having continuous scale does not depend on the approximate identity $\{e_n\}$ if A is separable (see 2.10).

2.6. **Example.** Let A be a separable simple AF C^*-algebra, p a nonzero projection in A and S be the set of traces τ on A such that $\tau(p) = 1$. Then S is a (weak*-) compact convex set. It is easy to verify that A has a continuous scale if and only if $\widehat{1}(\tau) = \tau(1)$ is continuous on S, where $\tau(1) = \limsup(\tau(e_n))$ and $\{e_n\}$ is an approximate identity consisting of projections. Therefore Definition 2.5 coincides with the definition in [11] or in [10]. In fact this is why we use the term "continuous scale."

2.7. **Example.** A simple C^*-algebra A is called purely infinite if for any two nonzero elements a and b in $P(A)$, $a \lesssim b$ (see [13] and [14]). It is clear that every σ-unital purely infinite simple C^*-algebra has a continuous scale. We show in [13] (see also [12, 19, 14, 15]) that the corona algebra $M(A)/A$ of every σ-unital purely infinite C^*-algebra is simple.

Theorem 2.8. *If A is a σ-unital simple C^*-algebra with continuous scale, then the corona algebra $M(A)/A$ is simple.*

Proof. Suppose that J is a closed ideal properly containing A. Choose a positive element $x \in J\backslash A$. As in [13, Theorem 2.3], we may assume that $y = \sum_{k=1}^{\infty} g_{2k}xg_{2k}$ is in $J^+\backslash A$, and $g_{2k}xg_{2k} \neq 0$, $0 \leq g_{2k}xg_{2k} \leq 1$. Since A has a continuous scale, we can choose an integer n_0 such that

$$\sum_{k=n_0}^{m} g_{2k} \lesssim f_{3/8}(g_2xg_2)$$

for all $m \geq n_0$. By induction, we can find a partition of $\{n_0+1, n_0+2, \ldots\}$ into finite subsets $N_1, N_2, \ldots$ (of consecutive integers) such that for each $n = 1, 2, \ldots$

$$\sum_{k\in N_n} g_{2k} \lesssim f_{3/8}(g_{2n}xg_{2n}).$$

For any $\varepsilon > 0$, there are $x_n \in P(A)$ such that

$$x_n \lessapprox f_{3/8}(g_{2n}xg_{2n}),$$

$$\left\| x_n - \sum_{k\in N_n} g_{2k} \right\| < \varepsilon/2^n,$$

$$\left\| x_n^{1/2} - \sum_{k\in N_n} g_{2k}^{1/2} \right\| < \varepsilon/2^n.$$

(See [13, 2.2].)

We may assume that $0 \leq x_n \leq 1$. It follows from [5, Lemma 1.7] that there are $z_n \in A$ such that $z_nz_n^* = x_n$, $z_n^*z_n \leq f_{1/8}(g_{2n}xg_{2n})$ and

$$z_n^*z_nf_{1/8}(g_{2n}xg_{2n}) = f_{1/8}(g_{2n}xg_{2n})z_n^*z_n = z_n^*z_n.$$

Hence $z_kz_j^* = 0$, if $k \neq j$. So

$$\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} z_k\right)^* = \sum_{k=1}^{n} z_kz_k^*$$

and $\|\sum_{k=1}^{n} z_kz_k^*\|$ is bounded. Thus $\{\|\sum_{k=1}^{n} z_k\|\}$ is bounded. It is then easy to see that $\sum_{k=1}^{n} z_k$ converges in the left strict topology to an element $z = \sum_{k=1}^{\infty} z_k$ in the left multiplier $\mathrm{LM}(A)$. To show that $\sum_{k=1}^{n} z_k$ converges strictly to z, it is enough to show that for each n, $g_n \sum_{k=N}^{\infty} z_k$ converges (in norm) to zero as $N \to \infty$. Write $z_k = (z_k z_k^*)^{1/2} u_k$. Then

$$\begin{aligned}
\left\| g_n \sum_{k=N}^{\infty} z_k \right\| &\le \left\| g_n \left(\sum_{k=N}^{\infty} z_k - \sum_{k=N}^{\infty} g_{2k}^{1/2} u_k \right) \right\| + \left\| g_n \sum_{k=N}^{\infty} g_{2k}^{1/2} u_k \right\| \\
&\le \left\| \sum_{k=N}^{\infty} x_k^{1/2} - \sum_{k=N}^{\infty} g_{2k}^{1/2} \right\| + \left\| g_n \sum_{k=N}^{\infty} g_{2k}^{1/2} u_k \right\| \\
&< \sum_{k=N}^{\infty} \varepsilon/2^k + \left\| g_n \sum_{k=N}^{\infty} g_{2k}^{1/2} u_k \right\| .
\end{aligned}$$

Since

$$g_n \left(\sum_{k=N}^{\infty} g_{2k}^{1/2} u_k \right) = 0$$

if $N > n+1$, we conclude that $\|g_n \sum_{k=N}^{\infty} z_k\| \to 0$ as $N \to \infty$, so $z \in M(A)$. Since

$$\begin{aligned}
z f_{1/8}(y) &= \left(\sum_{k=1}^{\infty} z_k \right) \left(\sum_{k=1}^{\infty} f_{1/8}(g_{2k} x g_{2k}) \right) \\
&= \left(\sum_{k=1}^{\infty} z_k \right) = z ,
\end{aligned}$$

$z \in J$. On the other hand

$$\left\| z z^* - \sum_{k \ge n_0} g_{2k} \right\| < \varepsilon .$$

Therefore $\sum_{k=1}^{\infty} g_{2k} \in J$, similarly $\sum_{k=1}^{\infty} g_{2k+1} \in J$. So $1 \in J$. This implies that $M(A)/A$ is simple.

Remark 2.9. If we examine the proof of 2.8 closely, we see that the same argument shows that if J is an ideal of $M(A)$ containing A properly, then $I \subset J$, the ideal defined in 2.2. So, if A is separable, by Lemma 2.4, I is the minimal ideal in $M(A)$ containing A properly. Therefore I does not depend on the choices of $\{e_n\}$.

Theorem 2.10. *Let A be a separable simple C^*-algebra. Then the corona algebra $M(A)/A$ is simple if and only if either A is elementary or A has a continuous scale.*

Proof. By Theorem 2.8, we need only to show the "only if" part. Suppose that $M(A)/A$ is simple and A is not elementary. It follows from Lemma 2.4 that $1 \in I$. Thus there is $a \in (I_0)^+$ such that

$$\|1 - a\| < \tfrac{1}{2} .$$

Therefore a is invertible. Hence for $m > n$

$$(e_m - e_n)^2 \sim (e_m - e_n)a^*a(e_m - e_n).$$

It follows from the definition of I_0 that A has a continuous scale.

3. ALGEBRAIC SIMPLICITY

In this section we will give more information about simple C^*-algebras with continuous scales.

Lemma 3.1. *If $a \gtrsim b$, $a, b \geq 0$, then for any $\varepsilon > 0$ there is $\delta > 0$, $x \geq 0$, and r such that*

$$f_\varepsilon(b) \leq rxr^* \lessapprox f_\delta(a).$$

In particular $f_\varepsilon(b) \tilde{<} f_\delta(a)$.

Proof. There is a sequence $\{s_k\}$ such that $s_k^* a s_k \to b$ in norm. For each n, there is $\delta_k > 0$ such that

$$\|s_k^* a s_k - s_k^* a f_{\delta_k}(a) s_k\| < 1/k.$$

Let $x_k = s_k^* a f_{\delta_k}(a) s_k$; then $x_k \geq 0$, $x_k \to b$. If $\|x_k - b\| < \varepsilon/2$, then, by [15, 2.2], there is r such that

$$f_\varepsilon(b) \leq r x_k r^* \lessapprox f_{\delta_k}(a).$$

Take $\delta = \delta_k$, $x = x_k$.

3.2. Let A be a simple C^*-algebra, $a \in P(A)_+$. We say that a is infinite, if there are $b, c \in P(A)_+$, $a \gtrsim b + c$, $b \perp c$, $b \neq 0$, $c \neq 0$ such that $b \gtrsim b + c$. An element a in $P(A)_+$ is said to be finite if it is not infinite. For nonpositive element a in $P(A)$, a is said to be infinite (finite) if a^*a is infinite (finite). In [14] we show that A is purely infinite if and only if every element in $P(A)$ is infinite.

Theorem 3.3. *If A is a σ-unital simple C^*-algebra with a continuous scale, then A is algebraically simple.*

Proof. We may assume that A is nonunital. Suppose that a is a strictly positive element of A, and $\{d_n\}$ is a sequence of decreasing positive numbers such that $d_n \to 0$, and $\{e_n = f_{d_n}(a)\}$ forms an approximate identity. Suppose that $x \in P(A)_+$. As in 2.4, there are $\{x_n\}$ and $\{y_n\}$ in $\mathrm{Her}(x)$ such that $0 \leq x_n \leq y_n \leq 1$, $y_n \perp y_m$ if $n \neq m$, $x_n y_n = y_n x_n = x_n$, and $\|x_n\| = 1$, $n = 1, 2, \ldots$.

Since A has a continuous scale, for each k, there is n_k such that for any $m > n_k$

$$(e_m - e_{n_k}) \lesssim x_k.$$

We may assume that $n_{k+1} > n_k$, $k = 1, 2, \ldots$.

Passing a subsequence, we may assume that $(e_m - e_n) \lesssim x_n$ if $m > n$ and $1 > d_n > d_{n+1}$, $n = 1, 2, \ldots$.

Set $t_n = (d_n - d_{n+1})/2$, let h_n be a continuous function on $[0, 1]$ defined by

$$h_n(t) = \begin{cases} 0, & t \geq d_n + t_n; \\ \text{linear}, & d_n < t < d_n + t_n; \\ 1, & d_{n+1} \leq t \leq d_n; \\ \text{linear}, & d_{n+2} + t_{n+1} < t < d_{n+1}; \\ 0, & t \leq d_{n+2} + t_{n+1}. \end{cases}$$

We have $h_{2n}(a) \lesssim x_n$, $h_{2n+1}(a) \lesssim x_n$, $n = 1, 2, \ldots$. It follows from Lemma 3.1 that

$$f_{1/4}(h_{2n}(a)) \mathrel{\tilde{<}} f_\delta(x_n),$$

and

$$f_{1/4}(h_{2n+1}(a)) \mathrel{\tilde{<}} f_\delta(x_n)$$

for some $\delta > 0$. There are $z_n \in A$ such that

$$z_n^* z_n = f_{1/4}(h_{2n}(a)) \quad \text{and} \quad z_n z_n^* \leq y_n.$$

Let $z = \sum_{n=1}^\infty (z_n/2^n)$. Since $z_n \perp z_m$ if $n \neq m$,

$$z^* z = \sum_{n=1}^\infty f_{1/4}(h_{2n}(a))/2^{2n},$$

$$z z^* \leq \sum_{n=1}^\infty y^n/2^{2n} \in \mathrm{Her}(x).$$

It follows from [15, 5.6.2] that $\mathrm{Her}(x) \subset P(A)$ and

$$\sum_{n=1}^\infty f_{1/4}(h_{2n}(a))/2^{2n} \in P(A).$$

Similarly

$$\sum_{n=1}^\infty f_{1/4}(h_{2n+1}(a))/2^{2n} \in P(A),$$

whence

$$b = \sum_{n=1}^\infty [f_{1/4}(h_{2n}(a)) + f_{1/4}(h_{2n+1}(a))]/2^{2n} \in P(A).$$

So $b = f(a)$, where f is a nonnegative continuous function on $\mathrm{sp}(a)$ such that $f(t) > 0$ for every $t \in \mathrm{sp}(a)$. Therefore b is a strictly positive element of A, since a is. By [15, 5.6.2], $A = \mathrm{Her}(b) \subset P(A)$. This completes the proof.

Corollary 3.4. *If A is a separable, nonelementary simple C^*-algebra with simple corona algebra $M(A)/A$, then A is algebraically simple.*

Remark 3.5. The converse of 3.4 is not true in general. Suppose that A is a separable AFC^*-algebra. B. Blackadar showed in [2] that A is algebraically simple if and only if A has a bounded scale ($\widehat{1}(\tau) = \tau(1)$ is bounded on S, see [11, 1]). It is clear that $\widehat{1}(\tau)$ is bounded does not imply $\widehat{1}(\tau)$ is continuous.

3.6. In 2.7, we show that every purely infinite simple C^*-algebra has a continuous scale. The following theorem shows that for stable C^*-algebras the converse is also true.

Theorem 3.7. *A σ-unital stable simple C^*-algebra A has a continuous scale if and only if it is purely infinite.*

Proof. We will show the "only if" part.

Let $\{b_n\}$ be an approximate identity for A and $\{e_{ij}\}$ be a matrix unit for K. Set $e_n = \sum_{i=1}^n b_n \otimes e_{ii}$. Then $\{e_n\}$ is an approximate identity for $A \otimes K$ $(\cong A)$. If $e_{n+2} - e_n \precsim e_1 = b_1 \otimes e_{11}$, since

$$e_{n+2} - e_n = b_{n+2} \otimes e_{n+2,n+2} + b_{n+1} \otimes e_{n+1,n+1} + \sum_{i=1}^{n} (b_{n+2} - b_n) \otimes e_{ii},$$

by 3.2, A has an infinite element. It follows from [3] that A contains a nontrivial projection p. By [4], $A \otimes K \cong pAp \otimes K$ $(\cong A)$. If $P(A)$ has a finite element $x \geq 0$, by the argument used in 3.3,

$$b = \sum_{i=k}^{\infty} \frac{1}{2^i} p \otimes e_{ii} \precsim x$$

for some k. By 3.2, since for every element $y \in \operatorname{Her}(b)$, $y \precsim b$, every element in $\operatorname{Her}(b)$ is finite. Put $q = \sum_{i=k}^{\infty} p \otimes e_{ii}$; then $q \in M(A)$ and qAq is stable. Moreover $qAq = \operatorname{Her}(b)$. So A $(\cong A \otimes K \cong \operatorname{Her}(b))$ has no infinite element, a contradiction. Therefore every element in $P(A)$ is infinite, so A is purely infinite (see [14]).

The following is a slight improvement of Theorem 3.2 in [16].

Theorem 3.8 (cf. [16, 3.2]). *Let A be a σ-unital stable C^*-algebra. Then the corona algebra $M(A)/A$ is simple if and only if either A is elementary or A is a purely infinite simple C^*-algebra.*

Proof. We may write $A = B \otimes K$ with B a σ-unital C^*-algebra. The "if" part follows from [13, Theorem 2.3]. The simplicity of the corona algebra $M(A)/A$ implies the simplicity of A follows from [17, 3.1]. Thus we may assume that A is simple. If A is nonelementary and (stably) semifinite, then the ideal I in [13, 2.6] is a closed ideal of $M(A)$ containing A properly (see [13, Lemma 2.7]). Let $b \in P(A)_+$ with $b \neq 0$; then $d(b \otimes e_{11}) \neq 0$. Since $\tilde{d}(1) \geq d(\sum_{i=1}^n b \otimes e_{ii}) = nd(b \otimes e_{11})$, $\tilde{d}(1) = \infty$ for each d (see [13, 2.7]), so $1 \notin I$. Therefore $M(A)/A$ is not simple. Thus A is not semifinite. It follows from [3] that A contains a nonzero projection p. By [4], $pAp \otimes K \cong A \otimes K \cong A$. So we may assume that B has a unital. Therefore Theorem 3.1 in [16] applies.

Remark 3.9. If we assume that A is separable, then 3.8 follows from 3.7 and from 2.9.

Errata. The "if" part of Theorem 2.8 in [13] needs assumptions that $\Delta(A)$ is unperforated and A has a cancelation property. However, 2.5, 2.6, 2.7, and the "only if" part of Theorem 2.8 in [13] are still true. For separable simple C^*-algebras, Theorem 2.9 in this paper is the correct version which includes non-stably-semifinite cases.

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Department of Mathematics, East China Normal University, Shanghai 200062, China