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RANDOM PERTURBATIONS OF SINGULAR SPECTRA

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ABSTRACT. The singular parts of the self-adjoint operators T and $H = T + V$ are mutually singular for "almost every" bounded perturbation V.

Let T and $H = T + V$ be self-adjoint operators on a *separable* Hilbert space $\mathscr{H}$. In 1965, Donaghue [1] proved the following theorem.

Theorem 1. *If $V = c\langle\cdot, \varphi\rangle\varphi$ is of rank one, and φ is T-cyclic, then the singular parts of T and H are supported on disjoint sets.*

We shall generalize this result by proving that the singular parts of T and $H = T + V$ are mutually singular for "almost every" bounded perturbation V. This theorem, which follows easily by the methods of Simon-Wolff [5] and the author [2], illustrates an essential instability of the singular spectrum.

Let T be self-adjoint, φ_n an *arbitrary* complete orthonormal set, $c_n > 0$ an arbitrary bounded sequence of positive numbers, and $X_n(\omega)$ a sequence of independent random variables, uniformly distributed on $[-1, 1]$. Define

$$H(\omega) = T + V(\omega),$$

where

$$V(\omega) = \sum_{n=1}^{\infty} c_n X_n(\omega)\langle\cdot, \varphi_n\rangle\varphi_n.$$

Theorem 2. *The singular parts T and $H(\omega)$ are supported on disjoint sets, almost surely.*

Proof. We sketch the proof, which follows [2]. Let N be a set of Lebesgue measure zero which supports the singular part of T. Define the multiplication operator

$$\mathbf{H}u(\omega) = H(\omega)u(\omega)$$

on $L_2(\Omega, P; \mathscr{H})$. □

Lemma. $\mathbf{H}$ *is spectrally absolutely continuous.*

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Assuming this for the moment, and taking $u(\omega) = u$ to be a constant function, we have

$$0 = |\mathbf{E}[N]u|^2 = \int_\Omega |E_\omega[N]u|^2 P(d\omega),$$

and hence

$$E_\omega[N]u = 0, \quad \text{a.s.}$$

Letting u range over a countable dense set gives

$$E_\omega[N] = 0, \text{ a.s.},$$

which is the result. (This argument is essentially due to Kotani.)

The following proof of the lemma is slightly more elementary than that of [2], so we have included it here.

Proof. Let Q be the coordinate operator of multiplication by x on $L^2(\mathbf{R})$ and

$$P = -i\frac{d}{dx}$$

its conjugate. If $f(t)$ is a bounded smooth function with $f'(t) > 0$ (for example, $\arctan(t)$), then because, under the Fourier transformation

$$Q = i\frac{d}{dp},$$

we have

$$i[f(P), Q] = f'(P) \geq 0.$$

If $\chi(x)$ is the characteristic function of $[-1, 1]$, then

$$i[\chi(Q)f(P)\chi(Q), \chi(Q)Q] = \chi(Q)f'(P)\chi(Q) > 0.$$

Thus, the operator $X = \chi(Q)Q$ of multiplication by x on $L^2[-1, 1]$ has a positive commutator with the bounded operator,

$$B = \chi(Q)f(P)\chi(Q).$$

Now, representing Ω as the infinite product of $[-1, 1]$'s, we have that

$$L^2(\Omega; \mathscr{H}) = \left(\bigotimes_{n=1}^{\infty} L^2[-1, 1]\right) \otimes \mathscr{H}.$$

Each $X_n(\omega)$ becomes multiplication by $\chi(Q)Q$ on one factor of the tensor product. Let B_n be as above with

$$i[B_n, X_n] > 0,$$

and define

$$\mathbf{A} = \sum_{n=1}^{\infty} \frac{1}{2^n} B_n.$$

Then

$$i[\mathbf{A}, \mathbf{H}] = i\sum_{n=1}^{\infty} c_n[B_n, X_n] \otimes \langle \cdot, \varphi_n \rangle \varphi_n \geq 0.$$

Hence, **H** has a positive commutator with a bounded operator of dense range and is therefore absolutely continuous by the Kato–Putnam theorem [4, p. 157]. □

Remarks. (1) One could replace the uniform distribution by other bounded, absolutely continuous distributions [2, p. 67]. In fact, as long as X_n are independent, one could presumably relax the requirement that they be identically distributed.

(2) One can take c_j tending very rapidly to zero, or not tending to zero at all, so that $V(\omega)$ is a very general sort of diagonal operator. This is rather satisfactory if one recalls that, by the Weyl-von Neumann Theorem [3, p. 523], every bounded self-adjoint operator differs from such an operator by an operator of arbitrarily small Hilbert-Schmidt norm.

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