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FULLY INDECOMPOSABLE EXPONENTS OF PRIMITIVE MATRICES

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ABSTRACT. If A is a primitive matrix, then there is a smallest power of A (its fully indecomposable exponent) that is fully indecomposable, and a smallest power of A (its strict fully indecomposable exponent) starting from which all powers are fully indecomposable. We obtain bounds on these two exponents.

1. INTRODUCTION

Let B_n denote the set of all matrices of order n over the Boolean algebra $\{0, 1\}$ where, in particular, $1 + 1 = 1$. Then B_n is a semigroup whose binary operation is ordinary matrix multiplication. Let J_n denote the matrix in B_n each of whose entries equals 1. A matrix $A \in B_n$ is *primitive* provided there is a positive integer k such that $A^k = J_n$; the least positive integer k satisfying $A^k = J_n$ is the *exponent* $e(A)$ of A. Exponents of primitive matrices have been well studied, and those numbers that are exponents of primitive matrices of order n have been completely determined [4, 6, 7]. In particular the largest exponent of a primitive matrix of order n is $e_n = n^2 - 2n + 2$. We denote the set of primitive matrices in B_n by P_n.

A matrix $A \in B_n$ is *partly decomposable* provided for some positive integers r and s with $r + s = n$, A has an r by s zero submatrix. A matrix in B_n that is not partly decomposable is called *fully indecomposable.* It is well known that a fully indecomposable matrix is primitive and the exponents of fully indecomposable matrices have been studied [3]. We denote the set of fully indecomposable matrices in B_n by F_n. Thus $F_n \subseteq P_n$. Moreover, it follows that

$$P_n = \{A\colon A \in B_n \text{ and } A^k \in F_n \text{ for some positive integer } k\}. \tag{1.1}$$

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There is a one-to-one correspondence between the set B_n and the set Γ_n of digraphs with vertex set $\{1, \dots, n\}$. This correspondence is given as follows: If $A = [a_{ij}] \in B_n$, then $\Gamma(A)$ is the digraph in which there is an arc (i, j) from i to j if and only if $a_{ij} = 1$ $(i, j = 1, \dots, n)$. We note that since A may have 1's on its main diagonal, the digraph $\Gamma(A)$ may have loops. We call a vertex i of a digraph a *loop-vertex* provided (i, i) is a loop of the digraph. We call a digraph in Γ_n a *primitive digraph* provided it corresponds to a primitive matrix. The following properties are direct consequences of the definitions:

(1.2) If A is primitive, $\Gamma(A)$ is *strongly connected* (that is, for each pair of distinct vertices i and j, there is a walk from i to j);

(1.3) A is primitive if and only if there is an integer k such that for each pair of distinct vertices i and j there is a walk in $\Gamma(A)$ of length k from i to j;

(1.4) [1] A^k is fully indecomposable if and only if for each set X of r vertices with $0 < r < n$, there are at least $r + 1$ different vertices that can be reached by a walk of length k which starts at a vertex in X.

In addition the following is a well-known characterization of primitive matrices:

(1.5) A is primitive if and only if the greatest common divisor of the lengths of all (elementary) cycles of $\Gamma(A)$ is 1.

Let A be a primitive matrix. It follows from (1.1) that there is a smallest positive integer k such that A^k is fully indecomposable; we denote the smallest such integer by $f(A)$. Schwarz [5] raised the question of determining the numbers

$$f_n = \max\{f(A) \colon A \in P_n\} \qquad (n \geq 1).$$

Before proceeding we observe an important difference that occurs in the investigation of the numbers $e(A)$ and $f(A)$ for primitive matrices A. If $A^k = J_n$, then because A can have no zero rows, $A^i = J_n$ for all $i \geq k$. However, if $A^k \in F_n$, then it does not necessarily follow that $A^i \in F_n$ for all $i \geq k$. For example, let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.6}$$

Then $A^i \notin F_7$ $(i = 1, \dots, 7)$; $A^8, A^9 \in F_7$; A^{10}, $A^{11} \notin F_7$; and $A^i \in F_7$ $(i \geq 12)$. (It follows (cf. (1.1)) that A is primitive since some power of A is

fully indecomposable.) For a primitive matrix A we define $f^*(A)$ to be the smallest positive integer k such that A^i is fully indecomposable for all $i \geq k$. We call $f(A)$ and $f^*(A)$, respectively, the *fully indecomposable exponent* and *strict fully indecomposable exponent* of the primitive matrix A. The matrix A in (1.6) satisfies $f(A) = 8$ and $f^*(A) = 12$. In general we have

$$f(A) \leq f^*(A) \leq e(A) \qquad (A \in P_n). \tag{1.7}$$

We define

$$f_n^* = \max\{f^*(A) : A \in P_n\} \qquad (n \geq 1), \tag{1.8}$$

and hereby raise the question of determining the numbers f_n^*. Clearly $f_n \leq f_n^*$ $(n \geq 1)$.

In what follows we obtain bounds for the numbers f_n and f_n^*.

2. Bounds on the exponents

Let A be a matrix in B_n with associated digraph $\Gamma(A)$, and let k be a nonnegative integer. For $X \subseteq \{1, \dots, n\}$, $R_k(X)$ denotes the set of all those vertices that can be *reached* by a walk of length k in $\Gamma(A)$ starting from a vertex in X. (If $k = 0$, $R_k(X) = X$.) A restatement of (1.4) is:

(2.1) A^k is fully indecomposable if and only if $|R_k(X)| > |X|$ for all X with $\phi \neq X \subsetneqq \{1, \dots, n\}$.

Chao and Zhang [2] showed that $f(A) \leq n$ if $A \in P_n$ and $\operatorname{trace}(A) \neq 0$. We refine this result to obtain an inequality for $f^*(A)$ in Theorem 2.2.

Lemma 2.1. *Let Γ be a strongly connected digraph with vertex set $\{1, \dots, n\}$, let s be a positive integer, and let $Z = \{i_1, \dots, i_s\}$ be a set of s loop-vertices of Γ. Then for each positive integer t,*

$$|R_t(Z)| \geq \min\{s + t, n\}. \tag{2.2}$$

Proof. Suppose that $R_t(Z) \neq \{1, \dots, n\}$. Since Γ is strongly connected, there is an arc (p, j) from some vertex $p \in R_t(Z)$ to a vertex $j \notin R_t(Z)$. Because $j \notin R_t(Z)$ there is a vertex in Z, say i_1, such that the distance from i_1 to p is t and the distance from each of $i_2, \dots, i_s$ to p is at least t. Thus there is a walk $(i_1, \dots, p)$ of length t from i_1 to p containing $t+1$ distinct vertices all of which are different from $i_2, \dots, i_s$. Since $i_1, i_2, \dots, i_s$ are loop-vertices, we conclude that

$$|R_t(Z)| \geq (s - 1) + (t + 1) = s + t,$$

and the lemma follows. □

Theorem 2.2. *Let s be a positive integer, and let A be a matrix in P_n having s 1's on its main diagonal. Then*

$$f^*(A) \leq n - s + 1. \tag{2.3}$$

Proof. Let Z be the set of s loop-vertices of the digraph $\Gamma(A)$, and let t be a positive integer. Let X be a set of vertices with $\phi \neq X \subsetneqq \{1, \dots, n\}$, and let $k = |X|$. We show that

$$|R_t(X)| \geq |X| + 1 \quad \text{for } t \geq n - s + 1. \tag{2.4}$$

If $|R_t(X)| = n$, then (2.4) holds. We now suppose that $|R_t(X)| < n$. First assume that $X \cap Z \neq \varnothing$. By Lemma 2.1

$$|R_t(X)| \geq |R_t(X \cap Z)| \geq |X \cap Z| + t.$$

Hence if $t \geq n - s + 1$,

$$|R_t(X)| \geq |X \cap Z| + n - s + 1 \geq |X| + 1.$$

Now assume that $X \cap Z = \varnothing$. Let x^* be a vertex in X and let z^* be a vertex in Z such that x^* has the minimum distance d to z^* among all vertices x in X and z in Z. Then

$$d \leq n + 1 - |Z| - |X| = n + 1 - s - k.$$

Because z^* is a loop-vertex, there are walks of length t from x^* to each vertex in $R_k(z^*)$ for each integer $t \geq (n + 1 - s - k) + k = n + 1 - s$. Using Lemma 2.1 we obtain that

$$|R_t(X)| \geq |R_t(\{x^*\})| \geq |R_k(z^*)| \geq k + 1$$

for each integer $t \geq n + 1 - s$. Hence (2.4) holds, and using (1.4) we conclude that (2.3) holds. □

It is easy to construct matrices A satisfying the hypotheses of Theorem 2.2 for which equality holds in (2.3). Let s and n be positive integers with $s \leq n$, and let Γ be the digraph obtained by including a loop at each of s consecutive vertices of a simple cycle of length n. Then the matrix $A \in B_n$ whose associated digraph is Γ is in P_n and satisfies $f^*(A) = n + 1 - s$.

Corollary 2.3 (cf. [2]). *Let the matrix A in P_n have nonzero trace. Then*

$$f(A) \leq f^*(A) \leq n. \tag{2.5}$$

Corollary 2.4. *Let A be a matrix in P_n. Suppose that the digraph $\Gamma(A)$ has a cycle of length r and that there are s vertices which belong to at least one cycle of length r. Then*

$$f(A) \leq r(n - s + 1). \tag{2.6}$$

Proof. The matrix A^r has s 1's on its main diagonal, and hence by Theorem 2.2 $(A^r)^{n-s+1}$ is fully indecomposable. □

The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

satisfies the hypotheses of Corollary 2.4 with $r = 3$ and $s = 4$. Hence by (2.6), $f(A) \leq 3$. Since $A^2 \neq J_4$, $f(A) = 3$ and equality holds in (2.6).

We note the following consequences of Corollary 2.4:

(2.7) If $\Gamma(A)$ has a Hamilton cycle (so $r = s = n$), then $f(A) \leq n$.

(2.8) If A is symmetric (so $r = 2$ and $s = n$), then $f(A) \leq 2$.

Corollary 2.5. *Let A be a matrix in P_n. Suppose that the digraph $\Gamma(A)$ has diameter d. Then $f(A) \leq 2d(n-d)$.*

Proof. Since $\Gamma(A)$ is strongly connected, there is a cycle of length $r \leq 2d$ containing $s \geq d+1$ distinct vertices. Hence by (2.6), $f(A) \leq 2d(n-d)$. □

We now use Corollary 2.4 to obtain an upper bound on the numbers $f_n\,(n \geq 1)$.

Theorem 2.6. $f_n \leq \lceil \frac{1}{4}(n-1)(n+3) \rceil$ *for* $n \geq 1$.

Proof. Let A be a matrix in P_n. Since $\Gamma(A)$ is strongly connected, $\Gamma(A)$ has a simple cycle of length r for some r with $1 \leq r \leq n$. Let s be the number of vertices which belong to at least one cycle of length r. Then $s \geq r$ and by Corollary 2.4,

$$f(A) \leq r(n-s+1) \leq r(n-r+1). \tag{2.9}$$

For integral r, $r(n-r+1)$ achieves its maximum when $r = (n+1)/2$ (n odd) and $r = n/2$, $n/2+1$ (n even). Since A is primitive, then by (1.5) the greatest common divisor of the lengths of the cycles is 1. Hence if n is odd and $r = (n+1)/2$, $\Gamma(A)$ has a cycle of length different from $(n+1)/2$. We now obtain from (2.6) that

$$f(A) \leq \begin{cases} (n^2+2n)/4, & (n \text{ even}) \\ (n^2+2n-3)/4, & (n \text{ odd}) \end{cases}$$

and the theorem follows. □

The upper bound for f_n in Theorem 2.6 probably is not of the same order of magnitude as f_n. The example (1.6) can be generalized to all $n \geq 5$ to show that $f_n \geq 2n-4$. Indeed it is tempting to *conjecture* that $f_n = 2n-4$ $(n \geq 5)$.

We now consider the maximum strict fully indecomposable exponent f_n^*. First we obtain by example a lower bound for f_n^*.

Suppose that k and n are integers with $n \geq 5$ and $2 \leq k \leq n-3$. Let A be the matrix in B_n whose associated digraph is pictured in Figure 1 (p. 1198). This digraph has cycles of lengths $n-k+1$ and $n-k$, and hence by (1.5) $A \in P_n$. Let $X_k = \{n-k+1, \ldots, n\}$. One easily checks that

$$|R_{i(n-k)-1}(X_k)| = i \qquad (i = 1, \ldots, k).$$

Hence it follows from (2.1) that

$$f^*(A) \geq k(n-k). \tag{2.10}$$

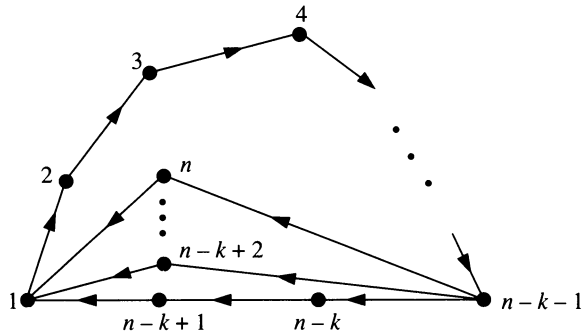

FIGURE 1. A digraph.

Indeed one may show that $f^*(A) = k(n-k)$. Taking $k = \lfloor n/2 \rfloor$ in (2.10) we obtain

$$f_n^* \geq \lfloor n/2 \rfloor \lceil n/2 \rceil \qquad (n \geq 5). \tag{2.11}$$

For a primitive matrix A in B_n, $f^*(A) \leq e(A) \leq e_n = n^2 - 2n + 2$ and hence it follows that

$$f_n^* \leq n^2 - 2n + 2. \tag{2.12}$$

Let $\lambda(A)$ denote the number of distinct lengths of the cycles of $\Gamma(A)$. It follows from (1.5) that $\lambda(A) \geq 2$ if $n > 1$. We now turn our attention to obtaining an improved bound for $f^*(A)$ in the case that $\lambda(A) = 2$.

Lemma 2.7. *Let Γ be a digraph in Γ_n, and let γ be a cycle of Γ of length r. If X is a set of vertices belonging to the cycle γ, then*

$$R_{ir+j}(X) \subseteq R_{(i+1)r+j}(X) \qquad (i \geq 0;\ 0 \leq j \leq r-1).$$

Proof. If from some x in X there is a path of length $ir + j$ to a vertex z, then there is also a walk from x to z of length $(i+1)r + j$. □

Lemma 2.8. *Let Γ be a digraph in Γ_n, and let γ be a cycle of Γ of length r. Let X be the set of all vertices belonging to the cycle γ. Then*

$$R_i(X) \subseteq R_{i+1}(X) \qquad (i \geq 0).$$

Proof. If from some vertex x of X there is a walk of length i to a vertex z, then there is a walk of length $i+1$ to z from the vertex which precedes x in Γ. □

Corollary 2.9. *If Z is the set of vertices of Γ consisting of those vertices z for which there is a walk of length at most d to z from some vertex of the cycle γ, then $R_d(X) = Z$.*

Lemma 2.10. *Let r and s be relatively prime positive integers with $r > s$. Let Γ be a digraph in Γ_n with exactly two cycles γ and μ where γ has length r,*

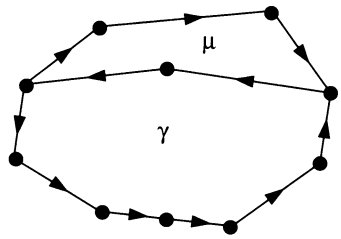

FIGURE 2. A digraph satisfying the hypotheses of Lemma 2.10 with $n = 10$, $r = 8$, and $s = 5$.

μ has length s, and γ and μ intersect. Let X be a nonempty set of vertices of the cycle γ. Then

$$|R_i(X)| \geq \min\{n, |X| + l\} \quad \text{if } i \geq lr \text{ and } l \geq 1. \tag{2.13}$$

Proof. A digraph Γ satisfying the hypotheses is pictured in Figure 2. Let Z be the set of vertices of γ, so that $\phi \neq X \subseteq Z$. For $i \geq 1$ let $X^{(i)}$ denote the set of vertices of γ that are reachable from X by a walk in γ of length i. It follows that $X^{(i)} = X^{(j)}$ if $i \equiv j \pmod r$. We first show that for $X \neq Z$, we have $|R_r(X) \cap Z| > |X|$. Since $X = X^{(r)} \subseteq R_r(X) \cap Z$, it suffices to show that $(R_r(X) \backslash X) \cap Z$ is nonempty. Suppose this set were empty. Then making use of the cycle μ we see that $X^{(r-s)} \subseteq R_r(X)$ and hence $X^{(r-s)} = X$. This implies that $X = X^{(s)}$ which contradicts the fact that r and s are relatively prime. It follows that (2.12) holds if $i = lr$ and $l \geq 1$. It now suffices to show that (2.12) holds for $l = 1$ and $r + 1 \leq i < 2r$.

If $X = Z$, then using Lemma 2.8 we see that

$$|R_i(X)| \geq \min\{n, |X| + i\}$$

for all $i \geq 1$. Hence we may assume that $X \neq Z$. If $R_i(X) \nsubseteq Z$, then the desired conclusion holds. We now assume that $R_i(X) \subseteq Z$. Contrary to what we wish to prove, we assume that $|R_i(X)| = |X|$, from which it follows that $R_i(X) = X^{(i)}$. Since $R_i(X) \subseteq Z$, we have that $R_{i-s}(X) \subseteq R_i(X)$. It follows that $X^{(i-s)} \subseteq X^{(i)}$ and hence that $X^{(i-s)} = X^{(i)}$, implying that the set $Y = X^{(i-s)}$ of vertices of γ satisfies $Y^{(s)} = Y$. Again we contradict the assumption that r and s are relatively prime. The proof of the lemma is now complete. □

We now obtain a bound for the strict fully indecomposable exponent $f^*(A)$ of a primitive matrix A for which the number $\lambda(A)$ of distinct cycle lengths of the digraph $\Gamma(A)$ equals 2.

Theorem 2.11. *Let A be a matrix in P_n satisfying $\lambda(A) = 2$. Then*

$$f^*(A) \leq \lfloor (n+1)^2/4 \rfloor.$$

Proof. Let the lengths of the cycles of $\Gamma(A)$ be r and s where $r > s$, and r and s are relatively prime. There exist cycles γ and μ of Γ with lengths r and s, respectively, such tht γ and μ intersect. Let the digraph Γ^* consist of the vertices and arcs of γ and μ, and let m be the number of vertices of Γ^*. Let Y be a subset of the vertices of Γ with $1 \leq k = |Y| \leq n-1$. First suppose that γ and μ above can be chosen so that Y contains $p \geq 1$ vertices of γ. Applying Lemma 2.10 and Corollary 2.9, we see that

$$|R_i(Y)| \geq k+1 \qquad (i \geq (k-p+1)r).$$

Since $r \leq n-(k-p)$, an easy calculation shows that

$$(k-p+1)r \leq \lfloor \tfrac{1}{4}(n+1)^2 \rfloor.$$

Now suppose that γ and μ cannot be chosen so that γ contains a vertex of Y. In this case $r \leq n-k$. There is a walk of length t from some vertex in Y to some vertex x of γ where $t \leq n-r-k+1$. Applying Lemma 2.10 and Corollary 2.9 again, we see that

$$|R_i(\{x\})| \geq k+1 \qquad (i \geq kr).$$

Hence

$$|R_i(Y)| \geq k+1 \qquad (i \geq kr+n-r-k+1).$$

An easy calculation now shows that

$$kr+n-r-k+1 \leq \lfloor n^2/4 \rfloor + 1.$$

It follows that for all Y with $\phi \neq Y \subsetneqq X$,

$$|R_i(Y)| \geq |Y|+1 \qquad (i \geq \lfloor (n+1)^2/4 \rfloor).$$

Hence A^i is fully indecomposable for all integers $i \geq \lfloor (n+1)^2/4 \rfloor$. □

We conclude with the following remark. Let A be a primitive matrix of order n. The exponent $e(A)$ of A satisfies $e(A) \leq n^2-2n+2$. If $\lambda(A) \geq 3$, then by a theorem of Lewin and Vitek [4], $e(A) \leq \lfloor (n^2-2n+2)/2 \rfloor + 1$. In particular, the largest exponent for matrices in P_n occurs among those matrices with $\lambda(A) = 2$. We conjecture that the strict fully indecomposable exponent behaves in a similar way and thus that

$$f_n^* \leq \lfloor (n+1)^2/4 \rfloor.$$

In view of (2.11) the validity of this latter inequality would imply that f_n^* is roughly $n^2/4$.

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