

## ON THE EXISTENCE OF WEAKLY $n$ -DIMENSIONAL SPACES

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**ABSTRACT.** Using a certain method for constructing peculiar large-dimensional spaces in every compactum with sufficiently large dimension, we present for every  $n$  an easy example of a weakly  $n$ -dimensional space.

### 1. INTRODUCTION

For a space  $X$  and a point  $x \in X$ ,  $\text{ind}_x X$  denotes the dimension of  $X$  at the point  $x$  (cf. [1, Problem 1.1.B]). If  $X$  is  $n$ -dimensional then its *dimensional kernel* is the set  $\{x \in X: \text{ind}_x X = n\}$ . It is known that the dimensional kernel of an  $n$ -dimensional space  $X$ , where  $n \geq 1$ , is an  $F_\sigma$  subset of  $X$  of dimension at least  $n - 1$ . Also, if in addition  $X$  is *compact*, then its dimensional kernel is  $n$ -dimensional. For more information see [1, Problem 1.5.C]. A space  $X$  is called *weakly  $n$ -dimensional*, where  $n \geq 1$ , if it is  $n$ -dimensional, but its dimensional kernel is of dimension  $n - 1$ . Clearly, a weakly  $n$ -dimensional space contains no compact subspace of dimension  $n$ . The first examples of weakly  $n$ -dimensional spaces were given by Sierpiński [15] and Mazurkiewicz [9]. A simpler construction can be found in Tomaszewski [16]; he also proved that if  $X$  is weakly  $n$ -dimensional and if  $Y$  is weakly  $m$ -dimensional then

$$\dim(X \times Y) \leq n + m - 1 = \dim X + \dim Y - 1.$$

In this note we present an application of a certain method for constructing peculiar large-dimensional subspaces in every compactum with sufficiently large dimension. The technique goes back to Mazurkiewicz [9] and Knaster [3] and has been used by several authors: see Lelek [8], Zarelua [17], Rubin, Schori, and Walsh [14], Kulesza [6], Ivanov [2], Pol [13], and Krasinkiewicz [4, 5] (These papers contain further references.) We use the technique for the construction of easy weakly  $n$ -dimensional spaces.

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## 2. PRELIMINARIES

Our terminology in dimension theory follows Engelking [1], Nagata [12], and van Mill [11].

We denote the closed interval  $[0, 1]$  by  $I$ . A *compactum* is a compact metrizable space. A family  $\tau = \{(A_1, B_1), (A_2, B_2), \dots, (A_n, B_n)\}$  of pairs of disjoint closed sets in a space  $X$  is called *essential* if for every family  $\{L_i: i \leq n\}$ , where  $L_i$  is a partition between  $A_i$  and  $B_i$  for every  $i$ , we have  $\bigcap_{i \leq n} L_i \neq \emptyset$ ; if  $\tau$  is not essential then it is called *inessential*. Recall that  $X$  is at least  $n$ -dimensional if and only if  $X$  contains an essential family of size  $n$ .

We shall need the following well-known fact.

**2.1. Lemma.** *Let  $K$  be a  $(k+1)$ -dimensional compactum and  $A, B$  a pair of disjoint closed subsets of  $K$ . Then there exists a continuous function  $f: K \rightarrow I$  taking  $A$  to 0 and  $B$  to 1 such that  $\dim f^{-1}(t) \leq k$  for each  $t \in [\frac{1}{3}, \frac{2}{3}]$ .*

This lemma follows immediately from Hurewicz's theorem (Kuratowski [7, §45]) that the zero-dimensional maps  $g: K \rightarrow I^{k+1}$  are dense in the function space  $C(K, I^{k+1})$  (if  $p: I^{k+1} \rightarrow I$  is the projection and  $g: K \rightarrow I^{k+1}$  is zero-dimensional then  $\dim[(p \circ g)^{-1}(t)] \leq k$  for all  $t \in I$ ). Alternatively, one can use Nagata's metric on  $K$  [12, Theorem V.4]. For the purpose of §2 it is enough to know that  $\dim f^{-1}(t) \leq k$  for  $t$  belonging to a Cantor set  $C$  in  $I$  and this can be proved directly by a standard Urysohn construction as the points  $t$  with  $\dim f^{-1}(t) \leq k$  form a set of type  $G_\delta$ .

We will also need the following triviality: if  $X$  is an  $n$ -dimensional  $\sigma$ -compact space, where  $n \geq 0$ , then there exists a zero-dimensional  $\sigma$ -compact set  $N \subseteq X$  such that  $\dim(X \setminus N) \leq n - 1$ . This can be verified easily by induction on  $n$ : note that  $\dim X = \text{ind } X$  and apply [11, Theorem 4.7.3].

The following lemma is probably well known: its easy proof is included for the sake of completeness.

**2.2. Lemma.** *Let  $X$  and  $Y$  be compact spaces, and let  $f: X \rightarrow Y$  be a continuous surjection. If  $Y$  is zero-dimensional at  $y \in Y$  and if  $\dim f^{-1}(y) = 0$  then  $X$  is zero-dimensional at every point of  $f^{-1}(y)$ .*

*Proof.* Pick an arbitrary  $x \in f^{-1}(y)$  and let  $U$  be a neighborhood of  $x$  in  $X$ . Since  $\dim f^{-1}(y) = 0$  there is an open and closed subset  $C$  of  $f^{-1}(y)$  such that  $x \in C \subseteq U$ . Pick disjoint open subsets  $E$  and  $F$  in  $X$  such that

$$E \cap f^{-1}(y) = C \quad \text{and} \quad F \cap f^{-1}(y) = f^{-1}(y) \setminus C.$$

It is clear that without loss of generality we may assume that  $E \subseteq U$ . Observe that  $E \cup F$  is a neighborhood of  $f^{-1}(y)$  in  $X$ . Consequently, since  $f$  is a closed map, there is a neighborhood  $V$  of  $y$  in  $Y$  such that  $f^{-1}(V) \subseteq E \cup F$ . Since  $Y$  is zero-dimensional at  $y$ , we may assume without loss of

generality that  $V$  is open and closed. Then  $f^{-1}(V) \cap E$  is an open and closed neighborhood of  $x$  in  $X$  which is contained in  $U$ .  $\square$

### 3. EASY WEAKLY $n$ -DIMENSIONAL SPACES

The aim of this section is to present new examples of weakly  $n$ -dimensional spaces.

**3.1. Theorem.** *Let  $K$  be an  $(n + 1)$ -dimensional compactum, where  $n \geq 1$ . Then  $K$  contains a weakly  $n$ -dimensional  $G_\delta$ -subset  $X$  (hence  $X$  is completely metrizable).*

*Proof.* Let  $\tau = \{(A_0, B_0), \dots, (A_n, B_n)\}$  be an essential family in  $K$ . By Lemma 2.1 there exists a continuous function  $f : K \rightarrow I$  taking  $A_0$  to 0 and  $A_1$  to 1 such that there exists a Cantor set  $\Delta \subseteq (0, 1)$  such that  $\dim f^{-1}(t) \leq n$  for every  $t \in \Delta$ .

We now closely follow a construction in Rubin, Schori, and Walsh [14]. Denote the hyperspace of  $K$  by  $2^K$  and put

$$\mathcal{E} = \{C \in 2^K : C \text{ is a continuum from } A_0 \text{ to } B_0\}.$$

Then  $\mathcal{E}$  is closed in  $2^K$  and hence is a compact space (see [11, Claim 1 of Theorem 4.7.10]). Consequently, there is a continuous surjection  $\varphi : \Delta \rightarrow \mathcal{E}$ . Put

$$Z = \bigcup \{f^{-1}(t) \cap \varphi(t) : t \in \Delta\}.$$

Then  $Z$  is closed in  $K$ , hence is compact, and  $f[Z] = \Delta$  ([11, Claim 2 of Theorem 4.7.10]). Then  $\dim Z \geq n$  because  $Z$  intersects every continuum from  $A_0$  to  $B_0$  ([11, Corollary 4.7.9]). In addition,  $\dim Z \leq n$  because the fibers of the restriction  $g$  of  $f$  to  $Z$  are at most  $n$ -dimensional and  $\Delta$  is zero-dimensional [1, Theorem 1.12.4]. We conclude that  $\dim Z = n$ .

Now let  $\Lambda$  denote the dimensional kernel of  $Z$ . Then because  $Z$  is compact,  $\Lambda$  is an  $n$ -dimensional  $\sigma$ -compact subset of  $Z$ , cf. the remarks in §2. There exists consequently a  $\sigma$ -compact zero-dimensional subset  $N$  of  $\Lambda$  such that  $\dim(\Lambda \setminus N) \leq n - 1$ , cf. the remark preceding Lemma 2.2. Now put  $X = Z \setminus N$ . Then  $X$  is clearly a  $G_\delta$ -subset of  $X$  and we claim that it is weakly  $n$ -dimensional.

Let  $C \in \mathcal{E}$ . We will prove that  $C$  meets  $X$ . Pick  $t \in \Delta$  such that  $\varphi(t) = C$ . Observe that  $g^{-1}(t) \subseteq C \cap Z$ . If  $\dim g^{-1}(t) > 0$  then  $g^{-1}(t)$  intersects  $X$  because the complement of  $X$  in  $Z$  is zero-dimensional. We may therefore assume that  $\dim g^{-1}(t) = 0$ . But since  $\Delta$  is zero-dimensional, we now obtain that  $Z$  is zero-dimensional at all points of  $g^{-1}(t)$  (Lemma 2.2). Consequently,  $g^{-1}(t) \cap \Lambda = \emptyset$ , that is,  $\emptyset \neq g^{-1}(t) \subseteq X$ . We conclude from [11, Corollary 4.7.9] that  $\dim X \geq n$ . However, because  $X$  is a subspace of  $Z$ , we also have  $\dim X \leq n$ . Consequently,  $\dim X = n$ , as required.

We next prove that  $X$  is weakly  $n$ -dimensional. This is however a triviality. Simply observe that if  $x$  is a point of  $X$  at which  $X$  is  $n$ -dimensional, then  $Z$  is  $n$ -dimensional at  $x$ , which implies that  $x \in X \cap \Lambda$ . Since by construction,  $\dim(X \cap \Lambda) \leq n - 1$ , we are done.  $\square$

3.2. *Remark.* Observe that the spaces  $X$  constructed in Theorem 3.1 are *rim-compact*, that is, have a basis consisting of open sets with compact boundaries. This is clear because the complement of  $X$  in  $K$  is zero-dimensional.

3.3. *Remark.* If one takes  $K = I^{n+1}$  in the above theorem then there is no need to use Lemma 2.1 in the proof. It seems that this gives us the easiest known examples of weakly  $n$ -dimensional spaces.

3.4. *Remark.* It seems useful to recall that the main points of the original construction of Mazurkiewicz [10]. Let  $C$  be the Cantor set and consider  $K = C \times I^n$ . Let  $\mathcal{F}$  denote the subspace of  $2^K \times 2^K \times \dots$  consisting of all sequences  $(A_1, A_2, \dots)$  such that  $A_1 \subseteq A_2 \subseteq \dots$ . Then  $\mathcal{F}$  is compact, so there is a continuous surjection  $\varphi: C \rightarrow \mathcal{F}$ . For  $t \in C$  we write  $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots)$ . Let  $\mathcal{E}_i = \{t \in C: \varphi_i(t) \cap (\{t\} \times I^n) \neq \emptyset\}$ ,  $D_1 = C_1$ ,  $D_{i+1} = \mathcal{E}_{i+1} \setminus \mathcal{E}_i$ , and let

$$M = \{(t, x): t \in D_i, x \in \varphi(t) \cap (\{t\} \times I^n)\}.$$

Then  $M \subseteq C \times I^n$  has the following property: for each  $S \subseteq M$  which projects onto  $C$ , each  $G_\delta$ -set  $G$  in  $C \times I^n$  containing  $S$ , also contains some section  $\{t\} \times I^n$ . This can be seen as follows. If  $(C \times I^n) \setminus G = F_1 \cup F_2 \cup \dots$ ,  $F_i \in 2^K$ ,  $F_1 \subseteq F_2 \subseteq \dots$ , then pick  $t \in C$  such that  $\varphi(t) = (F_1, F_2, \dots)$ . Then  $t \notin \mathcal{E}_i$  for all  $i$ , hence  $\{t\} \times I^n \subseteq G$ ; in particular,  $\dim S \geq n$ .

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