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AN INTEGRAL INEQUALITY

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ABSTRACT. We furnish conditions on the functions $p(t)$, $f(t)$, and $g(t)$ that are sufficient for the validity of the inequality, $\alpha^2\delta \geq \gamma^2\beta$, in which α, β, γ, and δ respectively, are the integrals over a measurable set E of $p(t)g(t)$, $p(t)g^2(t)$, $p(t)f(t)$, and $p(t)f^2(t)$.

1. INTRODUCTION

In this paper we consider conditions that assure the validity of the inequality,

$$\alpha^2\delta - \gamma^2\beta \geq 0, \tag{1}$$

in which

$$\alpha = \int_E p(t)g(t)\,dt, \qquad \beta = \int_E p(t)g^2(t)\,dt, \tag{2}$$
$$\gamma = \int_E p(t)f(t)\,dt, \qquad \delta = \int_E p(t)f^2(t)\,dt.$$

In spite of its simplicity and its resemblance to the Tchebycheff inequality [4, Theorem 10, p. 40], this inequality does not appear to have been discussed in the standard treatises on inequalities [2–4]. Our interest in (1) was triggered by the special case [6] in which $E = (0, 1)$, $p(t) = t$, $g(t) = J_0(vt)$, $f(t) = t^2 J_0''(vt)$, v is a fixed constant, and J_0 is the Bessel function of the first kind and order zero.

We always assume that E is a measurable set with positive measure, that the real-valued function $p(t)$ is integrable on E, and that the real-valued functions $g(t)$ and $f(t)$ are bounded and measurable on E. Consequently, the four integrals in (2) exist.

In §§2, 4 we prove the following two theorems, each of which states conditions sufficient for the validity of (1).

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Theorem 1. *The inequality* (1) *is true if*

$$p(s)p(t)p(u)[g^2(s)\{f(t)-f(u)\}^2+g^2(t)\{f(u)-f(s)\}^2 + g^2(u)\{f(s)-f(t)\}^2 - f^2(s)\{g(t)-g(u)\}^2 - f^2(t)\{g(u)-g(s)\}^2 - f^2(u)\{g(s)-g(t)\}^2] \geq 0 \tag{3}$$

a.e. on $E^3 = E\times E\times E$. *The inequality* (1) *is strict if* (3) *holds a.e. on* E^3 *and is strict on a measurable subset of* E^3 *whose measure is positive. Equality a.e. on* E^3 *in* (3) *implies that*

$$\alpha p(t)f(t) = \gamma p(t)g(t) \quad \text{a.e. on } E; \tag{4}$$

this condition implies equality in (1), *and in* (3) *if* $(\alpha, \gamma) \neq (0, 0)$.

Theorem 2. *The inequality* (1) *is true if*

$$p(t)p(u)\{g(t)f(u)-g(u)f(t)\}\{h(u)-h(t)\} \geq 0 \tag{5}$$

a.e. on $E^2 = E\times E$, *in which* $h(t) = \alpha f(t) + \gamma g(t)$. *The inequality* (1) *is strict if* (5) *holds a.e. on* E^2 *and is strict on a measurable subset of* E^2 *whose measure is positive. Equality a.e. on* E^2 *in* (5) *occurs if and only if either* (4) *holds or*

$$h(t) = \rho \quad \text{a.e. on } E - E_1, \tag{6}$$

in which E_1 *is the subset of* E *on which* $p(t)f(t) = 0$, $p(t)g(t) = 0$, *and* ρ *is a constant.*

Because it may be cumbersome to test inequality (3), in §3 we give a series of corollaries to Theorem 1 with progressively stronger hypotheses that are progressively easier to test. A similarly motivated corollary to Theorem 2 is given at the end of §4. We note that neither Theorem contains an explicit hypothesis on the sign of $p(t)$. Finally, in §5 we analyze the special case mentioned in the first paragraph. For this case we infer from Corollary 3 to Theorem 1 and from Corollary 4 to Theorem 2 that (1) holds strictly when $0 < v \leq v_1$ and when $0 < v \leq v_2$, respectively, in which $v_1 \sim 1.0944$ and $v_2 \sim 1.1668$. The first of these inferences is valid even if $p(t)$ is replaced by an arbitrary integrable and a.e. positive function.

2. Proof of Theorem 1

Fubini's Theorem and some simple manipulations show that

$$6(\alpha^2\delta - \gamma^2\beta) = 6\iiint_{E^3} p(s)p(t)p(u)g(s)f(u)\{g(t)f(u)-g(s)f(t)\}\,ds\,dt\,du = \iiint_{E^3} F(s,t,u)\,ds\,dt\,du,$$

in which

$$F(s,t,u) = 2p(s)p(t)p(u)\{g(s)g(t)f^2(u)+g(t)g(u)f^2(s)+g(u)g(s)f^2(t) - f(s)f(t)g^2(u) - f(t)f(u)g^2(s) - f(u)f(s)g^2(t)\}.$$

Because F is equal to the left side of (3), the assertions of Theorem 1 are now obvious, except for the assertions in the last sentence.

It is clear that (4) is a consequence of (3) when $(\alpha, \gamma) = (0, 0)$. Assume now that $(\alpha, \gamma) \neq (0, 0)$ and that equality occurs a.e. on E^3 in (3). Then

$$F(s, t, u) = 0 \quad \text{a.e. on } E^3, \ \alpha^2\delta - \gamma^2\beta = 0. \tag{7}$$

If we integrate the first equation in (7) with respect to t and u over E^2, we find that

$$p(s)\{\alpha^2 f^2(s) - \gamma^2 g^2(s) - 2\beta\gamma f(s) + 2\alpha\delta g(s)\} = 0 \quad \text{a.e. on } E. \tag{8}$$

If $\gamma = 0$, then $\alpha \neq 0$, $\delta = 0$, and (8) implies that $p(s)f^2(s) = 0$ a.e. on E; this is sufficient to guarantee (4). A similar argument disposes of the case when $\alpha = 0$, $\gamma \neq 0$. Henceforth we assume that $\alpha\gamma \neq 0$. Then it follows from the second equation in (7) that we may define ε as either $\alpha\delta/\gamma$ or $\gamma\beta/\alpha$. We then infer from (8) that

$$p(s)\{\alpha f(s) - \gamma g(s)\{\{\alpha f(s) + \gamma g(s) - 2\varepsilon\} = 0 \quad \text{a.e. on } E. \tag{9}$$

We define E_2 as the subset of E on which (9) holds and where $p(s)\{\alpha f(s) - \gamma g(s)\} \neq 0$. If $\operatorname{meas} E_2 = 0$, then (4) is surely true. If $\operatorname{meas} E_2 > 0$, if follows from (9) that

$$f(s) = \{2\varepsilon - \gamma g(s)\}/\alpha \quad \text{on } E_2, \tag{10}$$

$$p(s)f(s) = \gamma p(s)g(s)/\alpha \quad \text{a.e. on } E - E_2. \tag{11}$$

For almost all (s, t, u) in $E_2 \times (E - E_2) \times (E - E_2)$ the first equation in (7) implies, with the help of (10), (11), and a little algebra, that

$$p(s)p(t)g(t)p(u)g(u)\{\gamma g(s) - \varepsilon\}[\gamma\{g(t) + g(u)\} - 2\varepsilon] = 0.$$

Because $p(s)\{\gamma g(s) - \varepsilon\} = p(s)\{\gamma g(s) - \alpha f(s)\}/2 \neq 0$ on E_2 and $\operatorname{meas} E_2 > 0$, it follows that

$$p(t)g(t)p(u)g(u)[\gamma\{g(t) + g(u)\} - 2\varepsilon] = 0 \quad \text{a.e. on } (E - E_2)^2. \tag{12}$$

Now define E_3 and E_4 as the subsets of $E - E_2$ on which $p(t)g(t) > 0$ and $p(t)g(t) < 0$, respectively. First we consider the case in which $\operatorname{meas}(E_3 + E_4) = 0$. Then $p(t)g(t) = 0$ a.e. on $E - E_2$, it follows from (11) that $p(t)f(t) = 0$ a.e. on $E - E_2$, and we infer from (2) and (10) that

$$2\alpha\gamma = \int_{E_2} p(t)\{\alpha f(t) + \gamma g(t)\}\, dt = 2\varepsilon \int_{E_2} p(t)\, dt. \tag{13}$$

Therefore, $\varepsilon \neq 0$, and it follows from the first equation in (7) after some algebra that

$$\begin{aligned} &g^2(s)[\gamma\{g(t) + g(u)\} - 2\varepsilon] \\ &+ g(s)[\gamma\{g^2(t) - 6g(t)g(u) + g^2(u)\} + 2\varepsilon\{g(t) + g(u)\}] \\ &+ \gamma g(t)g(u)\{g(t) + g(u)\} - 2\varepsilon\{g^2(t) - g(t)g(u) + g^2(u)\} = 0 \end{aligned} \tag{14}$$

a.e. on E_2^3. Suppose, if possible, that

$$\gamma\{g(t) + g(u)\} - 2\varepsilon = 0 \quad \text{a.e. on } E_2^2. \tag{15}$$

An integration with respect to u over E_2 then shows that

$$g(t) = (2\varepsilon/\gamma) - (\operatorname{meas} E_2)^{-1} \int_{E_2} g(u)\, du \quad \text{a.e. on } E_2,$$

so that $g(t) = \varepsilon/\gamma$ a.e. on E_2, $f(t) = \varepsilon/\alpha$ a.e. on E_2, $\alpha f(t) = \gamma g(t)$ a.e. on E_2. From this contradiction of the definition of E_2, we infer that there is a subset D of E_2^2 such that $\operatorname{meas} D > 0$ and $\gamma\{g(t) + g(u)\} - 2\varepsilon$ is either positive, or is negative, a.e. on D. In either case, the result of an integration of (14) over D shows that $g(s)$ is a.e. on E_2 a solution of a quadratic equation, $Ag^2 + Bg + C = 0$, in which A, B, and C are constants such that $A \neq 0$. If this quadratic equation has unequal roots g_1 and g_2, and if the subsets E_5 and E_6 of E_2 on which $g(s) = g_1$ and $g(s) = g_2$, respectively, both have positive measure, then the first equation in (7) would imply, for almost all (s, t, u) in $E_5 \times E_6 \times E_6$, that $(g_1 - g_2)^2(\gamma g_2 - \varepsilon) = 0$, even though neither factor is zero. From this contradiction we infer that $g(s)$ is constant a.e. on E_2. It follows from (10) that $f(s)$ is also constant a.e. on E_2, and (4) is a consequence of the identity,

$$\begin{aligned} 0 &= \int_E p(s)\{\alpha f(s) - \gamma g(s)\}\, ds \\ &= (\alpha f - \gamma g) \int_{E_2} p(s)\, ds + \int_{E - E_2} p(s)\{\alpha f(s) - \gamma g(s)\}\, ds, \end{aligned}$$

equation (11), and the implication of (13) that

$$\int_{E_2} p(s)\, ds \neq 0.$$

Now consider the case in which $\operatorname{meas} E_3 > 0$ or $\operatorname{meas} E_4 > 0$. If we integrate (12) over E_3 or E_4, we find that $g(t)$ is a constant a.e. on $E_3 + E_4$. That constant must be ε/γ, $\varepsilon \neq 0$ by virtue of the definitions of E_3 and E_4. It then follows from (11) that $f(t) = \varepsilon/\alpha$ a.e. on $E_3 + E_4$. For almost all (s, t, u) in $(E_3 + E_4) \times E_2 \times E_2$, the first equation in (7) now shows, with the help of (10) and a little algebra, that

$$\{\gamma g(t) - \varepsilon\}\{\gamma g(u) - \varepsilon\}[\gamma\{g(t) - g(u)\} - 2\varepsilon] = 0.$$

Because the first two factors do not vanish on E_2, we see that (15) is true. A repetition of the argument following (15) leads to a contradiction, from which we infer that this case cannot occur.

This completes the proof that (4) is a consequence of equality a.e. on E^3 in (3). Conversely, it is obvious that equality in (1) is a consequence of (4) when $(\alpha, \gamma) = (0, 0)$, and almost as obvious that equality in both (1) and (3) is a consequence of (4) when $(\alpha, \gamma) \neq (0, 0)$. This completes the proof of Theorem 1.

3. Some corollaries to Theorem 1

We observe that the triple integral over E^3 of the left side of (3) is the same as

$$3 \iiint_{E^3} p(s)p(t)p(u)[g^2(s)\{f(t) - f(u)\}^2 - f^2(s)\{g(t) - g(u)\}^2]\, ds\, dt\, du.$$

The following corollary now follows at once from the proof of Theorem 1.

Corollary 1. *The inequality* (1) *is true if*

$$(16)\quad p(s)p(t)p(u)[g^2(s)\{f(t) - f(u)\}^2 - f^2(s)\{g(t) - g(u)\}^2] \geq 0 \quad \textit{a.e. on } E^3.$$

The inequality (1) *is strict if* (16) *holds a.e. on* E^3 *and is strict on a measurable subset of* E^3 *whose measure is positive. Equality a.e. on* E^3 *in* (16) *implies* (4), *and is implied by* (4) *if* $(\alpha, \gamma) \neq (0, 0)$.

We next state the following corollary.

Corollary 2. *Suppose that* $p(t) > 0$ *a.e. on* E, *that the subset of* E^2 *on which* $g(t) = g(u)$ *is a null set, and that*

$$(17)\quad \begin{aligned} & egl_{(t,u)\in E^2}[\{f(t) - f(u)\}^2/\{g(t) - g(u)\}^2] \\ & \qquad \geq \text{ess. l. u. b.}_{s\in E}[\{f(s)/g(s)\}^2]. \end{aligned}$$

Then the inequality (1) *is true, and is strict unless*

$$(18)\qquad \alpha f(t) = \gamma g(t) \quad \textit{a.e. on } E,$$

in which case (1) *is an equality, and* (17) *is an equality if* $(\alpha, \gamma) \neq (0, 0)$.

In order to prove Corollary 2, we first observe that $g(s) \neq 0$ a.e. on E and that the left side of (17) is finite. Because $p(t) > 0$ a.e. on E, it follows that (16) is a consequence of (17), and that the truth of (1) is assured by Corollary 1. Moreover, equality in (1) implies the existence of a constant μ such that

$$(19)\qquad \{f(s)/g(s)\}^2 = \mu = \{f(t) - f(u)\}^2/\{g(t) - g(u)\}^2 \quad \text{a.e. on } E^2.$$

A little algebra shows that $f(t)f(u) = \mu g(t)g(u)$ a.e. on E^2. Multiplication by $p(u)$ and integration with respect to u over E yields the result that $\gamma f(t) = \mu\alpha g(t)$ a.e. on E. A subsequent multiplication by $p(t)$ an integration with respect to t over E shows that $\gamma^2 = \mu\alpha^2$. The validity of (18) when $\gamma \neq 0$ is now obvious. If $\gamma = 0$, then either $\alpha = 0$, in which case (18) is surely true, or $\mu = 0$, in which case it follows from (19) that $f(s) = 0$ a.e. on E, so that (18) is surely true. Finally, it is clear when $(\alpha, \gamma) = (0, 0)$ that equality in (1) is a consequence of (18); this conclusion is almost as clear when $(\alpha, \gamma) \neq (0, 0)$.

Corollary 3. *Suppose that E is the closed interval (a, b), in which a and b are real numbers such that $a < b$, that $p(t) > 0$ a.e. on E, that $f(t)$ and $g(t)$ are continuously differentiable on E, that $g(t) \neq 0$ on E, that $g'(t) \neq 0$ when $a < t < b$, and that*

$$\min_{x\in E}[\{f'(x)/g'(x)\}^2] \geq \max_{s\in E}[\{f(s)/g(s)\}^2]. \tag{20}$$

Then the inequality (1) *is true, and is strict unless*

$$f(t) = (\gamma/\alpha)g(t) \quad on\ E. \tag{21}$$

Because $g'(t) \neq 0$ when $a < t < b$, the function $\phi(t, u)$ defined so that

$$\phi(t, u) = \{f(t - f(u)\}/\{g(t) - g(u)\} \quad \text{when } t \neq u,$$
$$\phi(t, u) = f'(u)/g'(u) \quad \text{when } t = u,$$

is lower semi-continuous on E^2. The function $f(s)/g(s)$ is continuous on E. We can now replace ess. g. l. b. and ess. l. u. b. in (17) with min and max. The inequalities (17) and (20) are equivalent because $\phi(t, u) = f'(x)/g'(x)$ for some x between t and u. Corollary 3 is now a consequence of Corollary 2 and the observation that $\alpha \neq 0$.

4. Proof of Theorem 2 and a corollary

The following calculation is sufficient to establish all of Theorem 2 except its last sentence.

$$\begin{aligned}
2(\alpha^2\delta - \gamma^2\beta) &= 2\alpha \int_E p(t)g(t)\,dt \int_E p(t)f^2(t)\,dt - 2\gamma \int_E p(t)f(t)\,dt \int_E p(t)g^2(t)\,dt \\
&= 2\alpha \left\{ \int_E p(t)g(t)\,dt \int_E p(u)f^2(u)\,du - \int_E p(u)f(u)\,du \int_E p(t)f(t)g(t)\,dt \right\} \\
&\quad + 2\gamma \left\{ \int_E p(t)g(t)\,dt \int_E p(u)f(u)g(u)\,du - \int_E p(u)f(u)\,du \int_E p(t)g^2(t)\,dt \right\} \\
&= 2 \iint_{E^2} p(t)p(u)g(t)f(u)\{h(u) - h(t)\}\,dt\,du \\
&= \iint_{E^2} p(t)p(u)\{g(t)f(u) - g(u)f(t)\}\{h(u) - h(t)\}\,dt\,du.
\end{aligned}$$

It is obvious when $(\alpha, \gamma) = (0, 0)$ that equality a.e. on E^2 in (5) is a consequence of either (4) or (6); this conclusion is almost as obvious when $(\alpha, \gamma) \neq (0, 0)$. The converse is obvious when $(\alpha, \gamma) = (0, 0)$. When $(\alpha, \gamma) \neq (0, 0)$ and equality occurs a.e. on E^2 in (5), an integration of (5) with respect to u over E shows that

$$p(t)g(t)\{\alpha\delta + \gamma\zeta - h(t)\} - p(t)f(t)\{\alpha\zeta + \gamma\beta - h(t)\} = 0 \tag{22}$$

a.e. on E, in which

$$\zeta = \int_E p(u)f(u)g(u)\,du.$$

When $\alpha = 0$, $\gamma \neq 0$ (or $\gamma = 0, \alpha \neq 0$), it follows from equality in (1) that $\beta = 0$ (or $\delta = 0$) and then from (22) that

$$p(t)\{\alpha f(t) - \gamma g(t)\}\{h(t) - \rho\} = 0 \quad \text{a.e. on } E, \tag{23}$$

if $\rho = \zeta$. When $\alpha\gamma \neq 0$, the same conclusion is valid if $\rho = \varepsilon + \zeta$, in which $\varepsilon = \alpha\delta/\gamma = \beta\gamma/\alpha$. We define E_7 as the subset of $E - E_1$ on which $h(t) \neq \rho$. If $\operatorname{meas} E_7 = 0$, then (6) is surely true, and if $\operatorname{meas}(E - E_1 - E_7) = 0$, then (4) is surely true. Henceforth, assume that $\operatorname{meas} E_7 > 0$, $\operatorname{meas}(E - E_1 - E_7) > 0$. We then infer from (23) that $\alpha f(t) = \gamma g(t)$ a.e. on E_7. If we multiply (5) as an equality, by α and then by γ, a little algebra shows that

$$p(t)g(t)\{h(t) - \rho\}p(u)\{\alpha f(u) - \gamma g(u)\} = 0,$$
$$p(t)f(t)\{h(t) - \rho\}p(u)\{\alpha f(u) - \gamma g(u)\} = 0$$

for almost all (t, u) in $E_7 \times (E - E_1 - E_7)$. Because $h(t) \neq \rho$ on E_7, $p(u) \neq 0$ on $E - E_1$, and $p(t)g(t)$ and $p(t)f(t)$ cannot both vanish at any point t in $E - E_1$, we conclude that $\alpha f(u) = \gamma g(u)$ a.e. on $E - E_1 - E_7$. Hence $\alpha f(u) = \gamma g(u)$ a.e. on $E - E_1$. This is sufficient to prove (4), and to complete the proof of Theorem 2.

Although the elementary concept of monotonicity for a single function is meaningless for a general set E, we can say that the pair of measurable functions $h(t)$ and $k(t)$ are "monotone in the same sense on E," or "similarly ordered," when

$$\{h(t) - h(u)\}\{k(t) - k(u)\} \geq 0 \quad \text{a.e. on } E^2. \tag{24}$$

This concept generalizes that defined in [4, p. 10] when E is an interval on the real line, and has been used in [3, p. 168].

Corollary 4. *Suppose that $p(t)g(t) > 0$ a.e. on E and that the functions $h(t) = \alpha f(t) + \gamma g(t)$ and $k(t) = f(t)/g(t)$ are monotone in the same sense on E. Then the inequality* (1) *is true and is strict if* (24) *is strict on a measurable subset of E^2 whose measure is positive. Finally, equality a.e. in* (24) *occurs if and only if either* (21) *holds a.e. on E or* (6) *holds a.e. on E.*

The inequalities (5) and (24) are equivalent, because $p(t)g(t) > 0$ a.e. on E. Moreover, $\alpha > 0$, the equalities (4) and (21) are equivalent, and $\operatorname{meas} E_1 = 0$. Now the corollary follows at once from Theorem 2.

5. A special case

If we rescale both the independent and the dependent variables in the special case mentioned in §1, it may be described as the case in which v is a positive number, $E = (0, v)$, $f(t) = t^2 J_0''(t)$, $g(t) = J_0(t)$, and $p(t)$ is integrable and positive a.e. on E. Then

$$f(t) < 0 \quad \text{if } 0 < t < j_1' \sim 1.8412, \tag{25a}$$

$$g(t) > 0 \quad \text{if } 0 < t < j_0 \sim 2.4048, \tag{25b}$$

$$g'(t) = -J_1(t) < 0 \quad \text{if } 0 < t < j_1 \sim 3.8317, \tag{25c}$$

in which j_n and j_n' are the smallest positive zeros [1, pp. 409, 411] of J_n and J_n', respectively, and J_n is the Bessel function of the first kind and order n. Moreover, $f'(t) = -t\{J_0(t) - tJ_1(t)\}$. Hence [1, p. 414] the smallest positive zero of $f'(t)$ is $v_m \sim 1.2558$, and

$$f'(t) < 0 \quad \text{if } 0 < t < v_m. \tag{25d}$$

It follows from (25) that $(f^2/g^2)' = 2f(f'g - fg')/g^3 > 0$ if $0 < t < v_m$, so that

$$\max_{t\in E}[\{f(t)/g(t)\}^2] = \{f(v)/g(v)\}^2 = \{v^2 J_0''(v)/J_0(v)\}^2 \tag{26}$$

if $0 < v \le v_m$. Moreover, $f'/g' > 0$ if $0 < t < v_m$ and

$$(f'/g')' = -2t - t\{J_1^2(t) - J_0(t)J_2(t)\}/J_1^2(t) < 0, \tag{27}$$

if $0 < t < j_1$, because [5, Equation (1), p. 152]

$$J_1^2(t) - J_0(t)J_2(t) = (8/t^2)\sum_{n=1}^{\infty} nJ_{2n}^2(t) > 0.$$

Therefore, $\{(f'/g')^2\}' = 2(f'/g')(f'/g')' < 0$ if $0 < t < v_m$, and

$$\min_{t\in E}[\{f'(t)/g'(t)\}^2] = \{f'(v)/g'(v)\}^2 = v^2[v - \{J_0(v)/J_1(v)\}]^2 \tag{28}$$

if $0 < v \le v_m$. It follows from (26) and (28) that (23) is true when $0 < v \le v_m$ if and only if $\{J_0^2(v) - J_1^2(v)\}\Phi(v) \ge 0$, in which $\Phi(v) = \{J_1(v)/J_0(v)\} + \{J_0(v)/J_1(v)\} - 2v$. We infer from the identity [5, Equation (14), p. 152],

$$v^2\{J_0^2(v) + J_1^2(v)\} = 4\sum_{n=0}^{\infty}(2n+1)J_{2n+1}^2(v),$$

that $J_0^2(v) > J_1^2(v)$ when $v^2 \le 2$, so that (23) is true when $0 < v \le v_m$ if and only if $\Phi(v) \ge 0$. We observe that $\lim_{v\to 0+}\Phi(v) = +\infty$, $\Phi(v_m) = (1 - v_m^2)/v_m < 0$, and

$$\Phi'(v) = -2 - \{J_0^2(v) - J_1^2(v)\}^3/\{J_0(v) + J_1(v)\}J_0(v)J_1(v)\}^2 < 0.$$

Therefore, (23) is true in the strict sense when $0 < v \le v_m$ if and only if $0 < v < v_1$, in which v_1 is the unique zero of $\Phi(v)$ such that $0 < v_1 < v_m$. We find that $v_1 \sim 1.0944$, and conclude from Corollary 3 that (1) holds when $0 < v \le v_1$.

We have not been able to use Theorem 1 or any of its corollaries to improve this result, even in the very special case when $p(t) = t$. On the other hand, it follows from (25) that $(f/g)' < 0$ if $0 < t < j_1$. Hence the hypotheses of Corollary 4 are satisfied if $h'(t) = \alpha f'(t) + \gamma g'(t) < 0$ when $0 < t < v < j_0$, or because $\alpha > 0$ and $g'(t) < 0$ when $0 < t < v < j_1'$, if $f'(t)/g'(t) > -\gamma/\alpha$

when $0 < t < v < j_1'$. It follows from (27) that this last condition holds if and only if

$$\Psi(v) = \{f'(v)/g'(v)\} + \int_0^v p(t)f(t)\,dt \Big/ \int_0^v p(t)g(t)\,dt \geq 0 \ .$$

It is easy to see that $\Psi(v_1) > 0 > \Psi(v_m)$ and that $\Psi'(v) = \{f'(v)/g'(v)\}' + \alpha^{-2}G(v)$ a.e. when $0 < v < v_m$, in which

$$G(v) = p(v)g(v)\int_0^v p(t)g(t)[\{f(v)/g(v)\} - \{f(t)/g(t)\}]\,dt\,.$$

Therefore, $\Psi'(v) < 0$ a.e. when $0 < v \leq v_m$, and there is a unique v_2 such that $v_1 < v_2 < v_m$ for which $\Psi(v_2) = 0$; moreover, $\alpha^2\delta - \gamma^2\beta > 0$ when $0 < v \leq v_2$.

When $p(t) = t$, the equation defining v_2 can be written in the form

$$J_0(v_2) - 2v_2J_1(v_2) + 3J_2(v_2) = 0\,.$$

We find that $v_2 \sim 1.1668$.

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