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HOMOTOPY-COMMUTATIVE H-SPACES

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ABSTRACT. Let X be an H-space with $H^*(X; Z_2) \simeq Z_2[x_1, \dots, x_d] \otimes \Lambda(y_1, \dots, y_d)$, where $\deg x_i = 4$ and $y_i = \mathrm{Sq}^1 x_i$. In this article we prove that X cannot be homotopy-commutative. Combining this result with a theorem of Michael Slack results in the following theorem: *Let X be a homotopy-commutative H-space with* mod 2 *cohomology finitely generated as an algebra. Then $H^*(X; Z_2)$ is isomorphic as an algebra over $A(2)$ to the* mod 2 *cohomology of a torus producted with a finite number of $CP(\infty)$s and $K(Z_{2^r}, 1)$s.*

0. INTRODUCTION

In this article we prove the following theorem:

Theorem A. *Let X be an H-space with*

$$H^*(X; Z_2) = Z_2[x_1, \dots, x_d] \otimes \Lambda(y_1, \dots, y_d),$$

where $\deg x_i = 4$ *and* $y_i = \mathrm{Sq}^1 x_i$. *Then X cannot be homotopy-commutative.*

The significance of Theorem A lies in its relationship to the following theorem, due to Michael Slack:

Theorem (Slack). *Let X be a homotopy-commutative H-space with* mod 2 *cohomology finitely generated as an algebra. Then*

(1) *All even-degree generators have infinite height and are in degrees two and four.*
(2) *All odd-degree generators lie in degrees one and five. The one-dimensional generators have infinite height and the five-dimensional generators are exterior.*
(3) $\mathrm{Sq}^1 : QH^4(X; Z_2) \to QH^5(X; Z_2)$ *is an isomorphism.*

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Combining Theorem A with Slack's theorem results in the following:

Theorem B. *Let X be a homotopy-commutative H-space with* mod 2 *cohomology finitely generated as an algebra. Then $H^*(X; Z_2)$ is isomorphic as an algebra over $A(2)$ to the* mod 2 *cohomology of a torus producted with a finite number of $CP(\infty)$s and $K(Z_{2^r}, 1)$s.*

This theorem extends work of Hubbuck and Lin [3, 4] which proves any finite homotopy-commutative H-space has the homotopy type of a torus. Here we treat the larger class of H-spaces whose cohomology is finitely generated as an algebra.

Theorem A also has the following immediate corollary:

Corollary C. *The 3-connective cover of S^3, $\widehat{S}^3$, is not* mod 2 *homotopy-commutative, hence neither is any product of $\widehat{S}^3$s homotopy-commutative.*

Remarks. (1) Corollary C can also be proved by using results in [8] and verifying that the Samelson product $\langle \alpha, \alpha \rangle \neq 0$, where α is a generator of $\pi_4(\widehat{S}^3)$. This approach was suggested to the authors by Chuck McGibbon. (The referee has observed that Corollary C can also be obtained quickly from results of Zabrodsky.)

(2) Our original proof of Theorem A rested upon an analysis of a partial Postnikov tower for X. The proof we give here is shorter and is based on a factorization of $\mathrm{Sq}^8\,\mathrm{Sq}^4$ (Theorem 1.1 below) that may well be of independent interest.

(3) In the rest of this paper, cohomology will be taken to have coefficients in Z_2 unless otherwise specified.

1. Factorization of $\mathrm{Sq}^8\,\mathrm{Sq}^4$

Theorem 1.1. *Let $x \in H^8(X; Z)$ be such that $\rho(x)$ lies in the kernel of Sq^2, where $\rho\colon H^*(X; Z) \to H^*(X; Z_2)$ is reduction* mod 2. *Then there is a formula*

$$\mathrm{Sq}^8\,\mathrm{Sq}^4 x = \mathrm{Sq}^4\,\tilde{\varphi}_{0,3}(x) + (\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})\varphi_{1,1}(x) + x(\mathrm{Sq}^4 x),$$

where the secondary operations $\tilde{\varphi}_{0,3}$ and $\varphi_{1,1}$ are defined by the following relations:

$$\tilde{\varphi}_{0,3} : \mathrm{Sq}^9 = (\mathrm{Sq}^7 + \mathrm{Sq}^{4,2,1})\,\mathrm{Sq}^2 \quad \textit{on an integral class}$$
$$\varphi_{1,1} : \mathrm{Sq}^2\,\mathrm{Sq}^2 = 0 \quad \textit{on an integral class}.$$

Proof. Consider the following two-stage Postnikov system.

$$\begin{array}{ccc} K(Z_2, 9) & & \\ \downarrow{\scriptstyle j} & & \\ E & & \\ \downarrow{\scriptstyle p} & & \\ K(Z, 8) & \xrightarrow{g} & K(Z_2, 10) \end{array}$$

in which $g^*(\iota_{10}) = \mathrm{Sq}^2 \iota_8$. Then there are elements $\tilde{v}_{0,3} \in H^{16}(E)$ and $v_{1,1} \in H^{11}(E)$ such that

$$j^*(\tilde{v}_{0,3}) = (\mathrm{Sq}^7 + \mathrm{Sq}^{4,2,1})\iota_9$$

and

$$j^*(v_{1,1}) = \mathrm{Sq}^2 \iota_9 .$$

Further, $\overline{\Delta}(\tilde{v}_{0,3}) = p^*(\iota_8) \otimes p^*(\iota_8)$, where $\overline{\Delta}$ denotes the reduced diagonal. We compute

$$j^*((\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})v_{1,1} + \mathrm{Sq}^4 \tilde{v}_{0,3}) = 0$$

and

$$\overline{\Delta}((\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})v_{1,1} + \mathrm{Sq}^4 \tilde{v}_{0,3}) = \overline{\Delta}(p^*(\iota_8)\,\mathrm{Sq}^4 p^*(\iota_8)) .$$

Hence

$$z = (\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})v_{1,1} + \mathrm{Sq}^4 \tilde{v}_{0,3} - p^*(\iota_8)\,\mathrm{Sq}^4 p^*(\iota_8)$$

is an element of $PH^{20}(E) \cap \ker(j^*)$. So $z \in p^* PH^{20}(K(Z, 8))$, and thus by [2] $z = c\,\mathrm{Sq}^{8,4} p^*(\iota_8)$, for some $c \in Z_2$.

To prove the theorem, we must show that $c = 1$. To do this, we loop the diagram four times, obtaining

$$\begin{array}{ccc} K(Z_2, 5) & & \\ \downarrow{\scriptstyle \tilde{j}} & & \\ \Omega^4 E & & \\ \downarrow{\scriptstyle \tilde{p}} & & \\ K(Z, 4) & \xrightarrow{\tilde{g}} & K(Z_2, 6) \end{array}$$

Since $\mathrm{Sq}^5 \iota_4 = \mathrm{Sq}^{2,1}\mathrm{Sq}^2 \iota_4$, there exists $\tilde{v}_{0,2} \in H^8(\Omega^4 E)$ such that $\tilde{j}^*(\tilde{v}_{0,2}) = \mathrm{Sq}^{2,1}\iota_5$ and $\overline{\Delta}(\tilde{v}_{0,2}) = \tilde{p}^*(\iota_4) \otimes \tilde{p}^*(\iota_4)$. So $\overline{\Delta}(\mathrm{Sq}^4 \tilde{v}_{0,2}) = \overline{\Delta}(\tilde{p}^*(\iota_4)^3)$. Thus

$$\begin{aligned} \mathrm{Sq}^4 \tilde{v}_{0,2} + \tilde{p}^*(\iota_4)^3 + (\sigma^*)^4(\tilde{v}_{0,3}) &\in PH^{12}(\Omega^4 E) \cap \ker(\tilde{j}^*) \\ &\subset \tilde{p}^* PH^{12}(K(Z, 4)) = 0 . \end{aligned}$$

We have

$$\begin{aligned} c\tilde{p}^*(\iota_4)^4 &= c\,\mathrm{Sq}^{8,4}\tilde{p}^*(\iota_4) \\ &= (\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})(\sigma^*)^4 v_{1,1} + \mathrm{Sq}^4(\sigma^*)^4\tilde{v}_{0,3} \\ &= \mathrm{Sq}^{7,2}(\sigma^*)^4 v_{1,1} + \mathrm{Sq}^4(\sigma^*)^4\tilde{v}_{0,3} \\ &= \mathrm{Sq}^{7,2}(\sigma^*)^4 v_{1,1} + \mathrm{Sq}^4\,\mathrm{Sq}^4\,\tilde{v}_{0,2} + \mathrm{Sq}^4\,\tilde{p}^*(\iota_4)^3 . \end{aligned}$$

Let $f\colon K(Z,2) \to K(Z,4)$ be defined by $f^*(\iota_4) = \iota_2^2$. Let $\hat{f}: K(Z,2) \to \Omega^4 E$ be a lifting of f. Then

$$\begin{aligned} c\iota_2^8 &= \hat{f}^*(\mathrm{Sq}^{7,2}(\sigma^*)^4 v_{1,1} + \mathrm{Sq}^4\,\mathrm{Sq}^4\,\tilde{v}_{0,2} + \mathrm{Sq}^4\,\tilde{p}^*(\iota_4)^3) \\ &= 0 + \lambda\,\mathrm{Sq}^{4,4}\iota_2^4 + \mathrm{Sq}^4\,\iota_2^6, \qquad \lambda = 0 \text{ or } 1 \\ &= \iota_2^8 . \end{aligned}$$

So $c = 1$. □

2. Relations in P_2X

Henceforth, we will assume that X is an H-space with mod 2 cohomology,

$$H^*(X) = Z_2[x_1, \dots, x_d] \otimes \Lambda(y_1, \dots, y_d)$$

as in the statement of Theorem A. For degree reasons, $H^*(X)$ is a primitively generated Hopf algebra.

According to [1], there is an exact triangle

$$\begin{tikzcd} H^*(P_2X) \arrow[rr, "\iota"] & & \overline{H}^*(X) \arrow[dl, "\overline{\Delta}"] \\ & \overline{H}^*(X) \otimes \overline{H}^*(X) \arrow[ul, "\lambda"] & \end{tikzcd} \tag{2.1}$$

in which ι has degree -1, λ has degree 2, and $\overline{\Delta}$ has degree 0.

Therefore there exist unique elements $u_i \in H^5(P_2X)$ and $v_i \in H^6(P_2X)$ such that $\iota(u_i) = x_i$, $\iota(v_i) = y_i$, and $\mathrm{Sq}^1 u_i = v_i$. A simple calculation following [1] shows that

$$\frac{Z_2[u_1, \dots, u_d, v_1, \dots, v_d, \mathrm{Sq}^4 u_1, \dots, \mathrm{Sq}^4 u_d]}{(\dots)^3} \subset H^*(P_2X). \tag{2.2}$$

Note that $\mathrm{Sq}^2 H^6(P_2X) = 0$. Using the Adem relation $\mathrm{Sq}^{2,3} = \mathrm{Sq}^5 + \mathrm{Sq}^{4,1}$ and the fact that $\mathrm{Sq}^3 u_i = 0$, we obtain

$$\mathrm{Sq}^4 v_i = \mathrm{Sq}^{4,1} u_i = \mathrm{Sq}^5 u_i = \mathrm{Sq}^{1,4} u_i .$$

Hence, if $\rho\colon H^*(P_2X; Z) \to H^*(P_2X; Z_2)$ is mod 2 reduction, we have

$$\mathrm{Sq}^4 H^6(P_2X; Z_2) \subset \mathrm{Im}(\rho). \tag{2.3}$$

Consider the following commutative diagram of infinite loop maps

$$
\begin{array}{ccc}
K(Z_2;7,9) & \longrightarrow & K(Z_2;7,9) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \bar{j}} \\
E & \xrightarrow{h} & \overline{E} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\
K(Z_2,5)\times K(Z,10) & \xrightarrow{\delta} & K(Z;6,10) \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \bar{w}} \\
K(Z_2;8,10) & \longrightarrow & K(Z_2;8,10)
\end{array}
\tag{2.4}
$$

in which

$$
\begin{aligned}
\bar{w}^*(\iota_8) &= \mathrm{Sq}^2\,\iota_6; & \bar{w}^*(\iota_{10}) &= \mathrm{Sq}^4\,\iota_6 - p(\iota_{10});\\
w^*(\iota_8)\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\iota_5; & & w^*(\iota_{10}) &= \mathrm{Sq}^4\,\mathrm{Sq}^1\,\iota_5 - p(\iota_{10});
\end{aligned}
$$

and

$$
\delta^*(\iota_{10}) = \iota_{10}; \qquad \delta^*(\iota_6) = \mathrm{Sq}^1\,\iota_5 .
$$

There exist elements $\bar{v}_{1,1} \in PH^9(\overline{E})$, $\bar{v}_{0,2} \in PH^{10}(\overline{E})$ such that $\bar{j}^*(\bar{v}_{1,1}) = \mathrm{Sq}^2\,\iota_7$ and $\bar{j}^*(\bar{v}_{0,2}) = \mathrm{Sq}^{2,1}\,\iota_7 + \mathrm{Sq}^1\,\iota_9$.

Lemma 2.1. *If* $v_{1,1} = h^*(\bar{v}_{1,1})$ *and* $v_{0,2} = h^*(\bar{v}_{0,2})$, *then* $\mathrm{Sq}^2\,v_{1,1} = \mathrm{Sq}^1\,v_{0,2}$.

Proof. Note that

$$
\mathrm{Sq}^2\,\bar{v}_{1,1} + \mathrm{Sq}^1\,\bar{v}_{0,2} \in \ker \bar{j}^* = \bar{p}^*(PH^{11}(K(Z;6,10))) = 0 .
$$

Therefore $\mathrm{Sq}^2\,\bar{v}_{1,1} = \mathrm{Sq}^1\,\bar{v}_{0,2}$. Applying h^* gives the lemma. □

Define $\varphi_{1,1}$ and $\varphi_{0,2}$ to be the secondary operations defined by the elements $v_{1,1}$ and $v_{0,2}$, respectively. We note that if $y \in H^5(P_2X)$, then $\varphi_{1,1}$ and $\varphi_{0,2}$ are defined on $\mathrm{Sq}^1\,y$ with zero indeterminacy.

We have the following relations:

(1) $\varphi_{1,1}\,\mathrm{Sq}^1\,H^5(P_2X) \subset \mathrm{Sq}^4\,H^5(P_2X)$, since $H^9(P_2X) = \mathrm{Sq}^4(P_2X)$.

(2) $\mathrm{Sq}^2\,\varphi_{1,1}\,\mathrm{Sq}^1\,H^5(P_2X) = \mathrm{Sq}^1\,\varphi_{0,2}\,\mathrm{Sq}^1\,H^5(P_2X)$, by Lemma 2.1.

Proposition 2.2. $\varphi_{0,2}\,\mathrm{Sq}^1\,H^5(P_2X) \subset \mathrm{Sq}^5\,H^5(P_2X)$.

Proof. Let y be in $H^5(P_2X)$. Then

$$\begin{aligned}\mathrm{Sq}^1\varphi_{0,2}\mathrm{Sq}^1 y &= \mathrm{Sq}^2\varphi_{1,1}\mathrm{Sq}^1 y \quad \text{by Lemma 2.1}\\ &= \mathrm{Sq}^2\mathrm{Sq}^4 y' \quad \text{by 1.}\\ &= \mathrm{Sq}^5\mathrm{Sq}^1 y'\\ &= \mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^1 y'\\ &= \mathrm{Sq}^1\mathrm{Sq}^1\mathrm{Sq}^4 y'\\ &= 0.\end{aligned}$$

But $H^{10}(P_2X)\cap\ker\mathrm{Sq}^1 \subset \mathrm{Sq}^5 H^5(P_2X)$. □

In the proofs of the next two theorems we will be using properties of the c-invariant and the H-deviation. We refer the reader to the appendix of [5] for a summary of general properties of these operations.

Theorem 2.3. $\varphi_{1,1}\mathrm{Sq}^1 H^5(P_2X) = 0$.

Proof. We now construct a Postnikov system which reflects the following relations for $u \in H^5(P_2X)$:

(1) $\varphi_{1,1}\mathrm{Sq}^1 u = \mathrm{Sq}^4 u'$,

(2) $\varphi_{0,2}\mathrm{Sq}^1 u = \mathrm{Sq}^5 u''$.

Consider the diagram

$$\begin{tikzcd}
 & K(Z_2;8,9) \arrow[d] & \\
 & E_1 \arrow[d] & \\
 & E \arrow[r, "w"] \arrow[d] & K(Z_2;9,10) \\
K(Z_2;5,5,5)\times K(Z;10,10) \arrow[rr, "w_0"] & & K(Z_2;8,10,10)
\end{tikzcd} \tag{2.5}$$

in which

$$w_0^*(\iota_8) = \mathrm{Sq}^{2,1}\iota_5, \qquad w_0^*(\iota_{10}) = \mathrm{Sq}^{4,1}\iota_5 - \iota_{10}, \qquad w_0^*(\iota'_{10}) = \mathrm{Sq}^{4,1}\iota'_5 - \iota'_{10},$$

and

$$w^*(\iota_9) = \varphi_{1,1}\mathrm{Sq}^1\iota_5 - \mathrm{Sq}^4\iota'_5, \qquad w^*(\iota_{10}) = \varphi_{0,2}\mathrm{Sq}^1\iota_5 - \mathrm{Sq}^5\iota''_5.$$

We shall retain the notation ι_5 and ι'_5 to denote the images of these elements in E and E_1.

In $H^*(E_1)$ we have

$$\begin{aligned}\mathrm{Sq}^6\, \iota_5' &= (\mathrm{Sq}^2\,\mathrm{Sq}^4 + \mathrm{Sq}^5\,\mathrm{Sq}^1)\iota_5' \\ &= \mathrm{Sq}^2\,\mathrm{Sq}^4\,\iota_5' + \mathrm{Sq}^1(\mathrm{Sq}^4\,\mathrm{Sq}^1\,\iota_5') \\ &= \mathrm{Sq}^2\,\varphi_{1,1}\,\mathrm{Sq}^1\,\iota_5' + \mathrm{Sq}^1\,\iota_{10}' \\ &= \mathrm{Sq}^1\,\varphi_{0,2}\,\mathrm{Sq}^1\,\iota_5 \\ &= \mathrm{Sq}^1\,\mathrm{Sq}^5\,\iota_5''.\end{aligned}$$

This Postnikov system can be delooped. It follows that there is an element $v \in H^{10}(E_1)$ with $j_1^*(v) = \mathrm{Sq}^2\,\iota_8 + \mathrm{Sq}^1\,\iota_9'$ and $\overline{\Delta} v = \iota_5' \otimes \iota_5'$. Therefore $c(\sigma^* v) = \sigma^*(\iota_5') \otimes \sigma^*(\iota_5')$. Since P_2X lifts to E_1, there is an H-lifting f_1 of X to ΩE_1. Checking degrees, one sees that X lifts to ΩE_1 by a c-map. This implies that

$$c(f_1^*\sigma^* v) = x' \otimes x' \neq 0, \quad \text{where } u' = i(x').$$

Therefore $f_1^*\sigma^* v$ is a nonzero primitive in $H^9(X)$. But $PH^9(X) = 0$. □

3. Proof of Theorem A

We can now prove Theorem A.

Proof. By (2.2), if $u \in H^5(P_2X)$, then $(\mathrm{Sq}^4 u)^2 \neq 0$. We have

$$\begin{aligned}(\mathrm{Sq}^4 u)^2 &= \mathrm{Sq}^{9,4} u \\ &= \mathrm{Sq}^{8,4,1} u \quad \text{since } \mathrm{Sq}^2 u = 0 \\ &= \mathrm{Sq}^4\,\tilde{\varphi}_{0,3}\,\mathrm{Sq}^1 u + (\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})\varphi_{1,1}\,\mathrm{Sq}^1 u \quad \text{by Theorem 1.1} \\ &= \mathrm{Sq}^4\,\tilde{\varphi}_{0,3}\,\mathrm{Sq}^1 u \quad \text{by Theorem 2.3}.\end{aligned}$$

Consider the following Postnikov system:

$$\begin{tikzcd}
K(Z_2; 8, 13) \arrow[d, "j_1"] & \\
E_1 \arrow[d, "p_1"'] & \\
E \arrow[r, "w_1"] \arrow[d, "r"] & K(Z_2; 9, 14) \\
K(Z_2, 5) \arrow[r, "w_0"] & K(Z_2; 7, 8)
\end{tikzcd}$$

in which

$$w_0^*(\iota_7) = \mathrm{Sq}^2\,\iota_5; \qquad w_0^*(\iota_8) = \mathrm{Sq}^{2,1}\,\iota_5;$$
$$w_1^*(\iota_9) = \varphi_{1,1}\,\mathrm{Sq}^1\,\iota_5; \qquad w_1^*(\iota_{14}) = \tilde{\varphi}_{0,3}\,\mathrm{Sq}^1\,\iota_5.$$

In $H^*(E_1)$,

$$\begin{aligned}(\mathrm{Sq}^4 \iota_5)^2 &= \mathrm{Sq}^{8,5}\iota_5 \\ &= \mathrm{Sq}^{8,4,1}\iota_5 \\ &= \mathrm{Sq}^4 \tilde{\varphi}_{0,3}\mathrm{Sq}^1 \iota_5 + (\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})\varphi_{1,1}\mathrm{Sq}^1 \iota_5 \\ &= 0.\end{aligned}$$

Therefore in $H^{16}(\Omega E_1)$ there exists an element v with

$$\begin{aligned}(\Omega j_1)^*(v) &= \mathrm{Sq}^4 \iota_{12} + (\mathrm{Sq}^9 + \mathrm{Sq}^{7,2})\iota_7, \\ \overline{\Delta} v &= \iota_4^2 \otimes \iota_4^2.\end{aligned}$$

We have the following commutative diagram

$$\begin{tikzcd}
 & K(Z_2;7,12) \arrow[d] & \\
 & \Omega E_1 & \\
 & \Omega E \arrow[u, "f_1"] \arrow[d, "r"] \arrow[r, "\Omega w_1"] & K(Z_2;8,13) \\
X \arrow[uur, "f_1"] \arrow[ur, "f_0"] \arrow[r, "f"] & K(Z_2,4) \arrow[r, "\Omega w_0"] & K(Z_2;6,7)
\end{tikzcd}$$

Here f is an H-map, since the relations defined by w_0 hold in P_2X. Further, $c(f)$ factors through $K(Z_2;4,5)$, so in fact f is a c-map. It follows that $(1-T^*)[D_{f_1}] = \Omega w_1 \circ c(f) \simeq *$. So $[D_{f_1}] \in H^{12}(X\Lambda X) \cap \ker(1-T^*)$.

Any terms of the form $x_i \otimes x_j^2 + x_j^2 \otimes x_i$ may be eliminated by changing f_1 by $x_i x_j^2$. It follows that

$$[D_{f_i}] = \sum_{j \neq k} b_{ijk}(x_i \otimes x_j x_k + x_j x_k \otimes x_i), \qquad b_{ijk} \in Z_2.$$

Therefore

$$\overline{\Delta} f_1^*(v) = x^2 \otimes x^2 + \mathrm{Sq}^4 \Sigma b_{ijk}(x_i \otimes x_j x_k + x_j x_k \otimes x_i) \neq 0.$$

This implies that if $\langle t, x^2 \rangle \neq 0$, then $t^2 \neq 0$ for $t \in H_*(X)$. But all squares are zero in $H_*(X)$, since $H_*(X)$ is primitively generated. We conclude that X is not homotopy-commutative. □

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