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A NOTE ON EIGENVALUES OF HECKE OPERATORS ON SIEGEL MODULAR FORMS OF DEGREE TWO

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ABSTRACT. Let F be a cuspidal Hecke eigenform of even weight k on $\mathrm{Sp}_4(\mathbb{Z})$ with associated eigenvalues λ_m $(m \in \mathbb{N})$. Under the assumption that its first Fourier-Jacobi coefficient does not vanish it is proved that the abscissa of convergence of the Dirichlet series $\sum_{m\geq 1} |\lambda_m| m^{-s}$ is less than or equal to k.

INTRODUCTION

Let F be a cuspidal Hecke eigenform of weight $k \in 2\mathbb{N}$ on $\mathrm{Sp}_4(\mathbb{Z})$ with associated eigenvalues λ_m $(m \in \mathbb{N})$ and assume that its first Fourier-Jacobi coefficient does not vanish. Then we shall show (§2) that the abscissa of convergence σ_0 of the Dirichlet series $\sum_{m\geq 1} |\lambda_m| m^{-s}$ is less than or equal to k. In an equivalent form, this means that $\sum_{1\leq m \leq N} |\lambda_m| = O_\varepsilon(N^{k+\varepsilon})$ for every $\varepsilon > 0$. Note that the (straightforward) estimate $\lambda_m = O(m^k)$ only implies $\sigma_0 \leq k+1$, while (for F not in the Maass space) the Ramanujan-Petersson conjecture would predict $\sigma_0 \leq k - \frac{1}{2}$.

Our result follows almost immediately from properties of Fourier-Jacobi coefficients of Siegel modular forms proved in [8]; however, it does not seem to have been noticed before.

1. ESTIMATES FOR EIGENVALUES

For k even denote by $S_k(\Gamma_2)$ the space of Siegel cusp forms of weight k on $\Gamma_2 := \mathrm{Sp}_4(\mathbb{Z})$ and write $T_k(m)$ $(m \in \mathbb{N})$ for the mth Hecke operator on $S_k(\Gamma_2)$. By definition,

$$T_k(m)F = m^{2k-3} \sum_{M \in \Gamma_2 \backslash O_2(m)} F|_k M \qquad (F \in S_k(\Gamma_2)),$$

where M runs through a set of representatives for the action of Γ_2 by left-multiplication on the set $O_2(m)$ of integral $(4, 4)$-matrices that are symplectic similitudes with scale m and as usual we have put

$$(F|_k M)(Z) = \det(CZ+D)^{-k} F((AZ+B)(CZ+D)^{-1})$$

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($M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $Z \in \mathfrak{H}_2 =$ Siegel upper half-space of degree 2).

Let $F \in S_k(\Gamma_2)$ be a nonzero eigenfunction of all Hecke operators, and denote by λ_m the eigenvalue of F with respect to $T_k(m)$. We let σ_0 be the abscissa of convergence of the Dirichlet series $\sum_{m\geq 1} |\lambda_m| m^{-s}$. Note that by classical results on Dirichlet series, one has $\sigma_0 = \inf\{\alpha \in \mathbb{R} | \sum_{1\leq m\leq N} |\lambda_m| = O(N^\alpha)\}$ (supposing that $\sum_{m\geq 1} |\lambda_m|$ diverges).

Counting the number of left cosets in $O_2(m)$ (cf. [1, 4, 9]) one easily obtains the estimate $\lambda_m = O(m^k)$ and hence $\sigma_0 \leq k+1$.

This can be improved to

$$\lambda_m = O_\varepsilon(m^{k-1/2+\varepsilon}) \qquad (\varepsilon > 0)$$

(hence $\sigma_0 \leq k + \frac{1}{2}$) if one combines on the one-hand side Kitaoka's estimate

$$a(T) = O_\varepsilon((\det T)^{k/2-1/4+\varepsilon}) \qquad (\varepsilon > 0),$$

where $a(T)$ is the Tth Fourier coefficient of F and T is any positive definite half-integral $(2, 2)$-matrix (cf. [6]), and on the other hand Andrianov's results relating the spinor zeta function $Z_F(s)$ of F to partial zeta functions of the form $\sum_{m\geq 1} a(mT)m^{-s}$ (cf. [1]); recall that if one puts $Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3} X^3 + p^{4k-6} X^4$ (p a prime), then $Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1} = \zeta(2s-2k+4)^{-1} \sum_{m\geq 1} \lambda_m m^{-s}$ [1]. Also by [1], $\widetilde{Z}_F(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$ has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \mapsto 2k-2-s$.

If F is in the Maass subspace $S_k(\Gamma_2)^* \subset S_k(\Gamma_2)$, then $Z_F(s) = \zeta(s-k+1) \times \zeta(s-k+2)L_f(s)$ where f is an elliptic cusp form of weight $2k-2$ [2]. From this one can easily conclude that $\lambda_m > 0$, and therefore $\sigma_0 = k$ for such F. Note that by [3, 10] one has $F \in S_k(\Gamma_2)^*$ iff $\widetilde{Z}_F(s)$ has poles.

On the other hand, for F in the orthogonal complement of the Maass space, one expects that the generalized Ramanujan-Petersson conjecture holds which predicts that the roots of the polynomial $Z_{F,p}(X)$ are of absolute value $p^{-k+3/2}$ for all p.

The Ramanujan-Petersson conjecture would imply the estimate

$$\lambda_m = O_\varepsilon(m^{k-3/2+\varepsilon}) \qquad (\varepsilon > 0)$$

and hence $\sigma_0 \leq k - \frac{1}{2}$, but at least at present a proof of it seems to be out of range.

2. Statement of result and proof

Recall that the function F has a Fourier-Jacobi expansion of the form

$$F(Z) = \sum_{m\geq 1} \varphi_m(\tau, z) e^{2\pi i m \tau'} \qquad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right),$$

where the φ_m's are Jacobi cusp forms of weight k and index m [2]. We shall prove:

Theorem. *Let* $F \in S_k(\Gamma_2)$ *be a nonzero Hecke eigenform with eigenvalues* λ_m *w.r.t.* $T_k(m)$ *and suppose that its first Fourier-Jacobi coefficient* φ_1 *does not vanish. Denote by* σ_0 *the abscissa of convergence of the Dirichlet series* $\sum_{m\geq 1} |\lambda_m| m^{-s}$. *Then* $\sigma_0 \leq k$.

Remark. The condition "$\varphi_1 \neq 0$" holds for (at least) all nonzero Hecke eigenforms of weight k with k in the range $10 \leq k \leq 32$ (cf. [12]). Being optimistic, one could hope that it is always satisfied (cf. [11]).

Proof of Theorem. The assertion is a more or less immediate consequence of the results proved in [8]. Let $J_{k,m}^{\text{cusp}}$ be the space of Jacobi cusp forms of weight k and index m and write $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{m,k}$ for the usual Petersson scalar product on $J_{k,m}^{\text{cusp}}$ [2]. Let G be any function in the Maass space $S_k(\Gamma_2)^*$ and write ψ_m for its mth Fourier-Jacobi coefficient. Then according to [5, 8] we have

$$\langle \varphi_m, \psi_m \rangle = \langle \varphi_1, \psi_1 \rangle \lambda_m.$$

Since $\varphi_1 \neq 0$ by assumption and the map $G \mapsto \psi_1$ is an isomorphism of $S_k(\Gamma_2)^*$ onto $J_{k,1}^{\text{cusp}}$ [2] we can find $G \in S_k(\Gamma_2)^*$ with $\langle \varphi_1, \psi_1 \rangle \neq 0$, and with this choice of G we can write

$$\lambda_m = \langle \varphi_m, \psi_m \rangle / \langle \varphi_1, \psi_1 \rangle.$$

By the Cauchy-Schwarz inequality we have

$$|\langle \varphi_m, \psi_m \rangle| \leq \|\varphi_m\| \cdot \|\psi_m\| \leq \tfrac{1}{2}(\|\varphi_m\|^2 + \|\psi_m\|^2).$$

On the other hand, as proved in [8], the Dirichlet series $\sum_{m\geq 1} \|\varphi_m\|^2 m^{-s}$ (resp. $\sum_{m\geq 1} \|\psi_m\|^2 m^{-s}$)—originally only defined in the half-plane $\mathrm{Re}(s) > k+1$—have meromorphic continuations to $\mathrm{Re}(s) \geq k$ with simple poles at $s = k$ as its only singularities. Therefore, by a theorem of Landau the abscissa of convergence of these Dirichlet series is equal to k. Hence we deduce $\sigma_0 \leq k$.

Remarks. (i) An interesting question seems to be whether the estimate $\|\varphi_m\|^2 = O_\varepsilon(m^{k-1+\varepsilon})$ would hold for any Hecke eigenform F in $S_k(\Gamma_2)$ (from the above discussion one sees that it is certainly satisfied for $F \in S_k(\Gamma_2)^*$). Note that under the condition "$\varphi_1 \neq 0$", this would imply that $\lambda_m = O_\varepsilon(m^{k-1+\varepsilon})$. To answer the above question one naturally seems to be led to a closer study of the Fourier-Jacobi coefficients of the Poincaré-type series introduced by Klingen in [7].

(ii) S. Böcherer informs the author that very recently J. S. Li (using results of R. Howe) proved the estimate $|\lambda_p| \leq 4p^{k-1}$ (p a prime).

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