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MAXIMAL FUNCTIONS, A_∞-MEASURES, AND QUASICONFORMAL MAPS

SUSAN G. STAPLES

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ABSTRACT. In the study of quasiconformal maps, one commonly asks, "Which classes of maps or measures are preserved under quasiconformal maps?", and conversely, "When does the said preservation property imply the quasiconformality of the map?". These questions have been studied by Reimann and Uchiyama with respect to the classes of BMO functions, maximal functions, and A_∞-measures. But, both authors assumed additional analytic hypotheses to establish the quasiconformality of the map. In this paper we utilize the geometry of quasiconformal maps to eliminate these auxiliary hypotheses and present results in the cases of maximal functions and A_∞-measures.

1. INTRODUCTION

In the study of quasiconformal mappings a commonly examined question is, "Which classes of maps or domains or measures are preserved under quasiconformal maps?". We can also examine the converse to this question and ask when the said preservation property implies the quasiconformality of the composition map.

This question has been studied by Reimann in relation to functions of bounded mean oscillation. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$ and let $u: D \to \mathbb{R}^n$ be a locally integrable function. We say that u is of *bounded mean oscillation* in D, $u \in \mathrm{BMO}(D)$, if $\|u\|_* < \infty$, where

$$\|u\|_* = \sup_{Q \subset D} |Q|^{-1} \int_Q |u(x) - u_Q| \, dx .$$

Here the supremum is taken over all open cubes $Q \subset D$, $|Q|$ is the Lebesgue measure of Q, and u_Q denotes the average of u over Q; namely,

$$u_Q = |Q|^{-1} \int_Q u(x) \, dx .$$

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Reimann proved a remarkable theorem on the relationship of quasiconformal mappings and the spaces $\mathrm{BMO}(D)$. We first recall the definition of absolutely continuous on lines, in order to understand the statement of his result.

Let $R = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$ be a closed n-interval. A mapping $\varphi: R \to \mathbb{R}^m$ is called absolutely continuous on lines, or ACL, if φ is continuous and if φ is absolutely continuous on almost every line segment in R, parallel to the coordinate axes. A mapping φ defined on D is called ACL if the restriction of φ on R is ACL for each closed n-interval R in D.

We now state Reimann's theorem.

1.1. **Theorem** [R, Theorems 2, 3]. *Let φ be a homeomorphism of D onto D'. Then the following two conditions are equivalent*:

(1.2) *φ is a quasiconformal mapping.*

(1.3) *φ is differentiable a.e. and ACL. In addition, the mapping $u \to u \circ \varphi$ is an isomorphism of the spaces* $\mathrm{BMO}(D')$ *and* $\mathrm{BMO}(D)$ *for which*

$$\tfrac{1}{c}\|u\|_{*,D'} \leq \|u \circ \varphi\|_{*,D} \leq c\|u\|_{*,D'}.$$

As quasiconformal maps are necessarily differentiable a.e. and ACL, we can ask whether these auxiliary hypotheses can be removed from the theorem of Reimann. These hypotheses are quite strong since they imply it is only necessary to establish the a.e. boundedness of the dilatation of the differential mapping, $d\varphi$, in order to prove quasiconformality. Reimann used this observation together with the analytic definition for quasiconformality given in terms of the Jacobian of φ. Because quasiconformal maps also can be defined geometrically, the differentiability assumptions may be unnecessary. If this were the case, then the preservation of BMO spaces would give a new defining property for quasiconformality.

Progress has been made by Astala [A] in examining this question. He showed that the differentiability and ACL assumptions can be removed from (1.3) if one assumes that

$$\tfrac{1}{c}\|u\|_{*,G'} \leq \|u \circ \varphi\|_{*,G} \leq c\|u\|_{*,G'} \tag{1.4}$$

holds for all subdomains $G \subset D$ and for all $u \in \mathrm{BMO}(G')$, where $G' = \varphi G$. Astala used Jones's [J] theorem on BMO extension domains along with geometric properties of uniform domains [GO] to arrive at his result.

Uchiyama has established analogues of Reimann's theorem using Hardy-Littlewood maximal functions and A_∞-measures in place of BMO functions. Again he assumed the additional hypotheses of differentiable a.e. and ACL in his theorems. The goal of this paper is to provide localized results for the cases of maximal functions and A_∞-measures analogous to the above mentioned result of Astala. We state each of Uchiyama's theorems followed by the corresponding localized theorems, which are the main results of this paper. Specifically, in §2 we examine maximal functions and quasiconformal maps. In §3 we first extend the concept of A_∞-measures on $\mathbb{R}^n$ to a larger class of measures, $A_\infty(D, \tau)$,

defined on any domain $D \subset \mathbb{R}^n$ and then provide new theorems relating quasiconformal maps and $A_\infty(D, \tau)$.

2. Maximal functions

We begin this section with a list of notation and a preliminary geometric observation. Here we assume that D and D' are domains in $\mathbb{R}^n$, where $n \geq 2$, and G and G' denote subdomains of D and D' respectively. By Q we denote any cube, and $Q(x)$ specifies that the cube is centered at x. Likewise, $B(x, r)$ denotes a ball of radius r centered at x and $B(r)$ or B_r denote the ball of radius r centered at the origin. We are often concerned with dilations of these balls and cubes, thus we define τQ, for $\tau \geq 1$, to be that cube with the same center as Q, and sidelength $\tau l(Q)$. Here $l(Q)$ is the sidelength of Q. We define $\sigma B(r)$ similarly. Further, $|Q|$ is the Lebesgue measure of Q and $d(Q, \partial D)$ is the distance from Q to the boundary of D.

Note the following elementary geometric fact for cubes $Q \subset D$,

$$\tau Q \subset D \quad \text{implies} \quad d(Q, \partial D) > (\tau - 1)/(2\sqrt{n}) \operatorname{diam} Q \tag{2.1}$$

and

$$d(Q, \partial D) > c \operatorname{diam} Q \quad \text{implies} \quad (2c + 1)Q \subset D. \tag{2.2}$$

Thus we adopt the simpler notation of $\tau Q \subset D$ in the theorems and proofs.

We define the Hardy–Littlewood maximal function $M^p_{\tau, D} f$ as follows

$$(M^p_{\tau, D} f)(x) = \sup_{\tau Q(x) \subset D} \left(|Q|^{-1} \int_Q |f(x)|^p \, dx \right)^{1/p}. \tag{2.3}$$

Here the supremum is taken over all cubes $Q(x)$ in D that satisfy $\tau Q(x) \subset D$. Note that when both $\tau = 1$ and $p = 1$, we have the standard Hardy–Littlewood maximal function, and whenever $\tau = 1$ or $p = 1$, we suppress them in the notation.

We state Uchiyama's Theorem and then directly follow with the statement of the new localized version of this theorem.

2.4. **Theorem** [U, Theorem 1]. *Let φ be a homeomorphism of D onto D'. Then the following are equivalent*:

(2.5) *φ is a quasiconformal mapping.*

(2.6) *φ is differentiable a.e. and ACL. Also, there exist constants c and p with $c > 0$ and $1 \leq p < \infty$ satisfying the following two inequalities*:

For all $x \in D$ there exists $r(x) > 0$ such that

$$\begin{aligned} &\sup \left\{ |Q|^{-1} \int_Q f(y)\, dy | x \in Q \text{ and } \operatorname{diam} Q < r(x) \right\} \\ &\quad \leq c \sup \left\{ \left(|Q|^{-1} \int_Q (f \circ \varphi^{-1}(y))^p \, dy \right)^{1/p} | \varphi(x) \in Q \subset D' \right\} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned}(2.8)\qquad &\sup\left\{|Q|^{-1}\int_Q f\circ\varphi^{-1}(y)\,dy \mid \varphi(x)\in Q \text{ and } \operatorname{diam} Q < r(x)\right\}\\ &\le c\sup\left\{\left(|Q|^{-1}\int_Q f(y)^p\,dy\right)^{1/p} \mid x\in Q\subset D\right\}\end{aligned}$$

for any nonnegative measurable function f *in* D. *Here note that the constraints on the cubes used in the suprema on the left satisfy* $B(x, r(x)) \subset D$ *and* $B(\varphi(x), r(x)) \subset D'$.

The main result of this section is the following:

2.9. **Theorem.** *Let* φ *be a homeomorphism of* D *onto* D'. *Then the following are equivalent*:

(2.10) φ *is a quasiconformal map.*

(2.11) *There exist constants* c, τ, *and* p *with* $c > 0$, $\tau \ge 1$, *and* $1 \le p < \infty$ *satisfying the following conditions for all subdomains* $G \subset D$ *and* $G' = \varphi(G) \subset D'$. *For all* $x \in G$ *and all nonnegative measurable functions* f *in* G,

$$(2.12)\qquad (M_{\tau,G}f)(x) \le c(M^p_{G'} f\circ\varphi^{-1})(\varphi(x))$$

and

$$(2.13)\qquad (M_{\tau,G'} f\circ\varphi^{-1})(\varphi(x)) \le c(M^p_G f)(x).$$

Before we commence the proof, we list the differences between Theorem 2.4 and Theorem 2.9. Primarily we have replaced the differentiability and ACL assumptions with a subdomain condition. Note also that in 2.9 the maximal functions at a point x are based at cubes centered at x, whereas in 2.4 this is not required. One positive gain from this change is that now the $r(x)$ condition of Uchiyama which varies with the point x is replaced with an understanding that the suprema on the left are taken over the class of cubes which can be expanded by a factor of τ and still remain in the subdomain.

Remark. Note that in Theorem 2.9, the role of τ is critical and the value of τ is closely tied to the dilatation of φ. To be specific, for example, it is not the case that inequalities (2.12) and (2.13) hold for $\tau = 1$ for every k-quasiconformal map φ.

For the proof that the quasiconformality implies preservation of maximal functions we follow the proof of Uchiyama with certain minor alterations. We rely on the following lemmata which are slight extensions of lemmata of Reimann and Gehring.

2.14. **Lemma.** *Let* $\varphi\colon D \to D'$ *be a* k*-quasiconformal mapping and let* $\sigma \ge 1$ *be given. Then there exists a constant* $\tau = \tau(k, n, \sigma)$ *such that to every cube* $Q(x) \subset D$ *with* $\tau Q \subset D$, *there exists a cube* $P'(\varphi(x)) \subset D'$ *that satisfies the following conditions*:

(2.15) $Q' = \varphi(Q) \subset P'$,

(2.16) $\sigma P' \subset D'$,
(2.17) $\operatorname{diam} P < d(P, \partial D)$, *where* $P = \varphi^{-1}(P')$,
(2.18) $|P| \leq b|Q|$, *where* $b = b(k, n, \sigma)$.

This lemma is a direct consequence of Lemma 4 in [R]. The strengthened statement in (2.16) depends only on a judicious choice of the constant τ to control the ratio s/r on p. 265 of the proof in [R].

2.19. **Lemma.** *Let $\varphi: D \to D'$ be a k-quasiconformal mapping. Then there exists a constant $\tau = \tau(k, n)$ such that for every cube $Q(x) \subset D$ with $\tau Q \subset D$, we have the following two conditions*:
(2.20) $\sqrt{n} \operatorname{diam} Q' < d(Q', \partial D')$ *where* $Q' = \varphi(Q)$,
and there exists a cube
(2.21) $P'(\varphi(x)) \subset D'$ *with* $\varphi^{-1}(P') = P \supset Q$ *and* $|P'| \leq c|Q'|$, *where* $c = c(k, n)$.

Condition (2.20) follows from condition (2.16) of Lemma 2.14. To ensure that the cube P' is contained in D' we need the constant $\sqrt{n}$ in (2.20). These are the only modifications required in the proof of Lemma 4 [G] to arrive at Lemma 2.19. The advantage of statements in the form of 2.14 and 2.19 is that now the geometric constraint, $\tau Q \subset D$, is based in terms of cubes in the original domain and not in terms of images of cubes in the image domain.

Proof of Theorem 2.9. We first assume that φ is k-quasiconformal, and let τ be the constant given by Lemma 2.19. Then for every cube $Q(x) \subset G$ with $\tau Q \subset G$, condition (2.20) gives

$$\operatorname{diam} Q' < \sqrt{n} \operatorname{diam} Q' < d(Q', \partial G').$$

Thus we can apply Gehring's integrability lemma, [G, Lemma 4]. Therefore there exist constants ε and c with $\varepsilon = \varepsilon(k, n) > 0$ and $c = c(k, n) > 0$ such that

$$\left(|Q|^{-1} \int_Q J_\varphi(x)^{1+\varepsilon}\, dx\right)^{1/1+\varepsilon} \leq c|Q|^{-1} \int_Q J_\varphi(x)\, dx.$$

Here $J_\varphi(x)$ is $|\det \phi(x)|$, where $\phi(x)$ is the Jacobian matrix of φ at x.

For the remainder of this proof, we denote by c any positive constant depending on n and k, not necessarily the same at every occurrence. Now from Coifman and Fefferman's paper on A_p weights, [CF, Theorem 5], there exist constants c and p with $c > 0$ and $1 < p < \infty$ such that

$$\left(|Q|^{-1} \int_Q J_\varphi(x)\, dx\right) \leq c \left(|Q|^{-1} \int_Q J_\varphi(x)^{-p'/p}\, dx\right)^{-p/p'}, \tag{2.22}$$

where $1/p + 1/p' = 1$.

Applying Hölder's inequality and using (2.22) and the change of variables

formula we obtain

$$\begin{aligned}|Q|^{-1}\int_Q f(y)\,dy &\le |Q|^{-1}\left(\int_Q J_\varphi(y)^{-p'/p}\,dy\right)^{1/p'}\left(\int_Q f(y)^p J_\varphi(y)\,dy\right)^{1/p}\\ &\le c\left(\int_Q J_\varphi(y)\,dy\right)^{-1/p}\left(\int_Q f(y)^p J_\varphi(y)\,dy\right)^{1/p}\\ &\le c\left(|\varphi(Q)|^{-1}\int_{\varphi(Q)}(f\circ\varphi^{-1}(y))^p\,dy\right)^{1/p}.\end{aligned}$$

By Lemma 2.19, there exists a cube $P'(\varphi(x)) \subset D'$ with $\varphi(Q) \subset P'$ such that $|P'| \le c|\varphi(Q)|$. Thus

$$|Q|^{-1}\int_Q f(y)\,dy \le c\left(|P'|^{-1}\int_{P'}(f\circ\varphi^{-1}(y))^p\,dy\right)^{1/p}.$$

This gives (2.12). Since φ^{-1} is also k-quasiconformal, (2.13) can be proven similarly.

We now prove that the subdomain condition in (2.11) implies the quasiconformality of φ. Once again we begin with a list of notation, definitions, and certain geometric observations.

We denote the cube Q that is centered at x and of sidelength $2r$ by $Q(x, r)$ and in the special case where $x = 0$, we write $Q(0, r) = Q(r)$. We let ω_n denote the n-dimensional Lebesgue measure of the unit ball.

Now let $\alpha = \tau\sqrt{n}$ and $\sigma = 2\alpha + 1$ and consider any ball B_r such that $\sigma B_r \subset D$. Then note that these choices of α and σ guarantee the following two geometric conditions:

$$\tau Q(r) \subset \alpha B_r \subset D \tag{2.23}$$

and

$$\tau Q(x, (\tfrac{1}{\alpha} + 1)r) \subset (2\alpha + 1)B_r \subset D \tag{2.24}$$

for all x satisfying $|x| = r$.

Recall the standard definition of the dilatation quotient $H(x)$ for φ. First we let

$$l(x, r) = \min_{|x-y|=r} |\varphi(x) - \varphi(y)| \quad\text{and}\quad L(x, r) = \max_{|x-y|=r} |\varphi(x) - \varphi(y)|.$$

We define $H(x)$ as

$$H(x) = \limsup_{r\to\infty} L(x, r)/l(x, r) = \limsup_{r\to 0} H(x, r).$$

We establish the quasiconformality of φ by showing that

$$H(x) \le a, \qquad a = a(n, c, \tau, p) \tag{2.25}$$

for all x in D, where c is the constant in (2.11).

Let $x \in D$ be fixed. We may assume without loss of generality that $x = \varphi(x) = 0$, and we procede to show $H(0) \leq a$. We write $L(r) = L(0, r)$ and $l(r) = l(0, r)$.

We define the region E as the union of all cubes centered at the origin which are contained in $\varphi(B_r)$; namely,

$$E = \bigcup_{Q(0) \subset \varphi(B_r)} Q(0). \tag{2.26}$$

Now consider the region $A = \varphi(B_r) \backslash E$, the subdomain $G = B_r$, and the function $f = x_A \circ \varphi$ at the point $x = 0$ in inequality (2.12). On the right-hand side we have

$$\sup_{Q(0) \subset G'} \left(|Q|^{-1} \int_Q (f \circ \varphi^{-1}(y))^p \, dy \right)^{1/p} = 0.$$

Inequality (2.12) then forces the situation that $\varphi^{-1}(A) \subset B(r) \backslash B(r/\alpha)$. In other words,

$$\varphi(B(r/\alpha)) \subset E. \tag{2.27}$$

Since we seek a uniform upper bound for $H(0, r)$, we may assume that $H(0, r) \geq 4\sqrt{n}$. Now choose x to be any point on ∂B_r such that $|\varphi(x)| = L(r)/2$. This time we consider $G' = D'$ and the function $f = x_E \circ \varphi$, where E is as in (2.26), in inequality (2.12) evaluated at the point x.

From relation (2.24) we see that we can use the cube $Q(x, (1 + \frac{1}{\alpha})r)$ on the left-hand side of (2.12). Combining this with observation (2.27) we have

$$\begin{aligned} \sup_{\tau Q(x) \subset G} |Q|^{-1} \int_Q (x_E \circ \varphi)(y) \, dy &\geq \frac{|Q(x, (1 + \frac{1}{\alpha})r) \cap \varphi^{-1}(E)|}{|Q(x, (1 + \frac{1}{\alpha})r)|} \\ &\geq \frac{(r/\alpha)^n \cdot \omega_n}{2^n r^n (1 + \frac{1}{\alpha})^n} = c_1 = c_1(\tau, n). \end{aligned} \tag{2.28}$$

We also can make estimates for the right-hand side of (2.12) by noting that any cube $Q(\varphi(x))$ that intersects E satisfies

$$l(Q(\varphi(x))) \geq \frac{2}{\sqrt{n}} \left(\frac{L(r)}{2} - l(r)\sqrt{n} \right) \geq \frac{L(r)}{2\sqrt{n}},$$

since by assumption, $H(0, r) \geq 4\sqrt{n}$.

Thus

$$\begin{aligned} c_1 &\leq c \sup_{Q(\varphi(x)) \subset G'} \left(|Q|^{-1} \int_Q (x_E)^p(y) \right)^{1/p} \\ &\leq c \left(\frac{|E|}{(L(r)/2\sqrt{n})^n} \right)^{1/p} \leq c \left(\frac{4n(\omega_n)^{1/n} \cdot l(r)}{L(r)} \right)^{n/p}. \end{aligned}$$

Thus we obtain

$$H(0, r) = L(r)/l(r) \leq c_2 = c_2(n, \tau, p, c).$$

Finally, since the condition $\sigma B_r \subset D$ holds for all $r \le r_0 = d(0, \partial D)/\sigma$, we can conclude that

$$H(0) = \limsup_{r \to 0} \frac{L(r)}{l(r)} \le a = \max(4\sqrt{n}, c_2),$$

thus establishing the quasiconformality of φ. □

3. A_∞-MEASURES

In this section we examine the relationship between A_∞-measures and quasiconformal maps. We generalize Uchiyama's result for $A_\infty(\mathbb{R}^n)$ to the case for $A_\infty(D)$. For this result we need the following definition.

When a measure μ is defined on the Borel sets of a domain $D \subset \mathbb{R}^n$, we say that $\mu \in A_\infty(D, \tau)$, $\tau \ge 1$, if there exist $c > 0$ and $\delta > 0$ such that both

$$\mu(E)/\mu(Q) \le c(|E|/|Q|)^\delta \tag{3.1}$$

and

$$|E|/|Q| \le c(\mu(E)/\mu(Q))^\delta \tag{3.2}$$

hold for all cubes Q satisfying $\tau Q \subset D$ and all measurable $E \subset Q$. When $\tau = 1$ we write $A_\infty(D, 1) = A_\infty(D)$.

Note that just as is the case with $A_\infty(\mathbb{R}^n)$ (see [CF]), conditions (3.1) and (3.2) are equivalent, that is either one can be used to define $A_\infty(D, \tau)$. For the ease of computations later, we assume that they both hold with the same constants c and δ.

We first recall Uchiyama's theorem before stating the new results.

3.3. **Theorem** [U, Theorem 2]. *Let φ be a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$. Then the following are equivalent*:

(3.4) *φ is quasiconformal.*

(3.5) *φ is differentiable a.e. and ACL. Furthermore, $\mu(\varphi^{-1}(\cdot))$ and $\mu(\varphi(\cdot))$ are A_∞-measures for all $\mu \in A_\infty$.*

We begin by describing what happens to the class $A_\infty(D')$ under composition with a quasiconformal map $\varphi: D \to D'$.

3.6. **Theorem.** *Let $\varphi: D \to D'$ be a k-quasiconformal map, let G' be any subdomain of D, and let $\mu \in A_\infty(G')$ with associated constants c and δ. Then $\nu = \mu \circ \varphi \in A_\infty(G, \tau)$, where $G = \varphi^{-1}(G')$ and ν has associated constants $c' = c'(c, \delta, k, n)$, $\delta' = \delta'(\delta, k, n)$, and $\tau = \tau(k, n)$.*

We need the following result of Reimann.

3.7. **Lemma** [R, Corollary, p. 262]. *Let φ be a k-quasiconformal map from D to D', and let Q be any cube in D such that* $\operatorname{diam}(Q') < d(Q', \partial D')$, *where $Q' = \varphi(Q)$. Then*

$$|E'|/|Q'| \le c(|E|/|Q|)^\delta, \qquad c = c(k, n), \ \delta = \delta(k, n)$$

for every measurable set $E \subset Q$ with image $E' = \varphi(E)$.

Proof of Theorem 3.6. Apply Lemma 2.14 with $\sigma = 2\sqrt{n} + 1$ so that for each cube $Q(x) \subset G$ with $\tau Q(x) \subset G$ we get a cube $P' \subset G'$, which satisfies $\operatorname{diam} P' < d(P', \partial G')$ as well as $\operatorname{diam} P < d(P, \partial G)$ and $|P| \le b|Q|$. Let this τ be the τ in Theorem 3.6. Henceforth in this proof we continue with the notation that Q is any cube in G such that $\tau Q \subset G$, P' is the associated cube in G' from Lemma 2.14, and $Q' = \varphi(Q)$, $P = \varphi^{-1}(P')$.

Since we will work with cubes in G' to invoke the hypothesis that $\mu \in A_\infty(G')$ and subsequently deal with cubes in G, we first seek to find an upper bound of the type

$$\mu(P')/\mu(Q') \le B, \qquad B = B(k, n, c, \delta). \tag{3.8}$$

By hypothesis we have

$$|Q'|/|P'| \le c(u(Q')/\mu(P'))^{\delta}. \tag{3.9}$$

Next since $\operatorname{diam} P < d(P, \partial G)$, we can apply Lemma 3.7 to φ^{-1} to get

$$|Q|/|P| \le c_1(|Q'|/|P'|)^{\delta_1}, \qquad c_1 = c_1(k, n), \ \delta_1 = \delta_1(k, n). \tag{3.10}$$

Combining (2.18), (3.9), and (3.10) yields

$$\frac{1}{b} \le \frac{|Q|}{|P|} \le c_1(c^{\delta_1}) \left(\frac{\mu(Q')}{\mu(P')}\right)^{\delta\delta_1}$$

so that (3.8) is established.

Now consider any measurable set $E \subset Q$ and write $E' = \varphi(E)$. Condition (3.8) coupled with the hypothesis that $\mu \in A_\infty(G')$ yields

$$\frac{\mu(E')}{\mu(Q')} \le B\frac{\mu(E')}{\mu(P')} \le Bc\left(\frac{|E'|}{|P'|}\right)^{\delta} \le Bc\left(\frac{|E'|}{|Q'|}\right)^{\delta}.$$

We apply Lemma 3.7 again since we have $\operatorname{diam} Q' \le \operatorname{diam} P' < d(P', \partial G') \le d(Q', \partial G')$, and arrive at

$$\frac{\mu(\varphi(E))}{\mu(\varphi(Q))} = \frac{\mu(E')}{\mu(Q')} \le Bcc_1^{\delta_1}\left(\frac{|E|}{|Q|}\right)^{\delta\delta_1}.$$

This proves Theorem 3.6. □

We shortly state and prove the converse to Theorem 3.6. Because we apply Astala's theorem [A] and thus must verify that the same constant c holds for all subdomains in condition (1.4), we take careful note of the interdependence of constants. We also make use of the duality between BMO and A_∞ noted by Moser [M] and Reimann [R] as well as the equivalent definitions for A_∞ given by Coifman and Fefferman [CF].

We begin by stating these background lemmata on A_∞ giving only brief mention if any of how the original proofs need to be modified to arrive at the $A_\infty(D, \tau)$ form.

3.11. **Lemma** [CF, Theorem 4 and Lemma 5]. *The following two conditions are equivalent for* $d\mu = w(x)\,dx$ *a measure on* D.

(3.12) $\mu \in A_\infty(D, \tau)$.

(3.13) *There exist constants* $r > 0$ *and* $q > 0$ *such that for all cubes* Q *with* $\tau Q \subset D$,

$$|Q|^{-1} \int_Q w(x)\,dx \le r \left(|Q|^{-1} \int_Q w(x)^{-q}\,dx \right)^{-1/q}.$$

Here (3.12) implies (3.13) with $r = r(c, \delta)$ and $q = q(c, \delta)$ and similarly (3.13) implies (3.12) with $c = c(r, q)$ and $\delta = \delta(r, q)$.

3.14. **Lemma** [M, Lemma 3]. *Let* $f \in \mathrm{BMO}(D)$. *Then the measure* μ *given by*

$$\mu(G) = \int_G e^{\varepsilon f(x)}\,dx \quad \text{where } G \subset D \text{ and } \varepsilon = \frac{a}{\|f\|_*}, \quad a = a(n) \tag{3.15}$$

is in $A_\infty(D)$ *with constants* $c = c(n)$ *and* $\delta = \delta(n)$.

Here, note that a, c, and δ are constants that are determined by the John-Nirenberg Theorem [JN] and depend only on the dimension n.

3.16. **Lemma.** *Let* μ *be a measure in* $A_\infty(D, \tau)$ *with associated constants* c *and* δ *for some* $\tau \ge 1$ *and denote* $d\mu = w(x)\,dx$. *Then* $f(x) = \log w(x)$ *is in* $\mathrm{BMO}(D)$ *and* $\|f\|_*$ *is bounded above by a number that depends only on* c, δ, τ, *and* n.

Proof. By Lemma 3.11, we can rewrite the A_∞ hypothesis as condition (3.13) and apply Lemma 3 in [R] to arrive at

$$\sup_{\tau Q \subset D} |Q|^{-1} \int_Q |f - f_Q|\,dx \le a, \qquad a = a(c, n, \delta).$$

Now we make use of the following inequality

$$\sup_{Q \subset D} |Q|^{-1} \int_Q |f - f_Q|\,dx \le c_0 \sup_{\tau Q \subset D} |Q|^{-1} \int_Q |f - f_Q|\,dx,$$

(see [RR, Lemma 2] or [S, Corollary 2.26]), to conclude that

$$\|f\|_* \le c_0 a = c_1 = c_1(c, \delta, n, \tau). \quad \square$$

At this point we make a comment on the notation $A_\infty(D, \tau)$. From the above lemmata we can deduce the following corollary.

3.17. **Corollary.** *Let* $\mu \in A_\infty(D, \tau)$, *with associated constants* c *and* δ, *where* $d\mu = w(x)\,dx$. *Then there exists a constant* $\varepsilon = \varepsilon(\tau, c, \delta, n)$ *such that* $\mu' \in A_\infty(D)$, *where* $d\mu' = w(x)^\varepsilon\,dx$.

Now we are ready to state and to prove the analogue of Uchiyama's Theorem 2.

3.18. **Theorem.** *Let $\varphi\colon D \to D'$ be a homeomorphism. Assume that there exists a constant $\tau \geq 1$ such that for all subdomains G and G' with $G' = \varphi(G)$ both of the following conditions hold*:

(3.19) *If $\mu \in A_\infty(G')$ with associated constants c and δ, then $\nu = \mu \circ \varphi \in A_\infty(G, \tau)$, where the A_∞-constants for ν satisfy $c' = c'(c, \delta, \varphi)$ and $\delta' = \delta'(c, \delta, \varphi)$.*

(3.20) *If $\mu \in A_\infty(G)$ with associated constants c and δ, then $\nu = \mu \circ \varphi^{-1} \in A_\infty(G', \tau)$ with associated constants $c' = c'(c, \delta, \varphi)$ and $\delta' = \delta'(c, \delta, \varphi)$.*

Then φ is quasiconformal.

Proof. We modify the ideas in Uchiyama's proof in order to apply Astala's theorem. Choose any function $f \in \mathrm{BMO}(G')$. Then by Lemma 3.14 the measure μ defined by

$$\mu(E) = \int_E e^{\varepsilon f(x)}\, dx$$

is in $A_\infty(G')$ with associated constants $c = c(n)$, $\delta = \delta(n)$, and $\varepsilon = a/\|f\|_*$, $a = a(n)$. By hypothesis, the measure $\nu = \mu \circ \varphi$ defined by

$$\nu(E) = \mu \circ \varphi(E) = \int_E e^{\varepsilon f(\varphi(x))} J_\varphi(x)\, dx$$

is in $A_\infty(G, \tau)$ with associated constants $c' = c'(c, \delta, \varphi) = c'(n, \varphi)$ and $\delta' = \delta'(c, \delta, \varphi) = \delta'(n, \varphi)$.

Now by Lemma 3.16, $f' = \log e^{\varepsilon f(\varphi(x))} J_\varphi(x) \in \mathrm{BMO}(G)$ with

$$\|f'\|_* \leq b, \qquad b = b(c', \delta', \tau, n) = b(\tau, n, \varphi). \tag{3.21}$$

Let μ be the Lebesgue measure on G' and note that μ is in $A_\infty(G')$. We can apply our hypothesis to this μ to see that the measure φ^* given by $\varphi^*(E) = |\varphi(E)|$ is in $A_\infty(G, \tau)$. Thus $\log J_\varphi(x) \in \mathrm{BMO}(G)$ by Lemma 3.16 where

$$\|\log J_\varphi(x)\|_* \leq B, \qquad B = B(\tau, n, \varphi). \tag{3.22}$$

Combining (3.21) and (3.22), we conclude that

$$\|f \circ \varphi\|_* \leq \varepsilon^{-1}(b + B) = m_1 \|f\|_*, \qquad m_1 = m_1(\tau, n, \varphi). \tag{3.23}$$

Similarly by using (3.20), we can deduce

$$\|f \circ \varphi\|_* \geq m_2 \|f\|_*, \qquad m_2 = m_2(\tau, n, \varphi). \tag{3.24}$$

It now follows from Astala's theorem [A], that φ is quasiconformal. □

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TEXAS 78712