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SOME TRACE CLASS COMMUTATORS OF TRACE ZERO

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ABSTRACT. It is shown that if T is an operator on a separable complex Hilbert space and X is a Hilbert-Schmidt operator such that $TX - XT$ is a trace class operator, then the trace of $TX - XT$ is zero provided one of the two conditions holds: (a) T^2 is normal; (b) T^n is normal for some integer $n > 2$ and $T^*T - TT^*$ is a trace class operator. Related results involving essentially unitary operators and Cesàro operators are also given.

1. INTRODUCTION

In their work on the traces of commutators of integral operators, Helton and Howe [8, Lemma 1.3] proved that if A is a selfadjoint operator on a separable complex Hilbert space and X is a compact operator such that the commutator $AX - XA$ is in the trace class, then the trace of $AX - XA$ is equal to zero. In [18, Lemma 8] Weiss proved that if N is a normal operator and X is a Hilbert-Schmidt operator such that $NX - XN$ is in the trace class, then the trace of $NX - XN$ is also zero. These results have been extended to certain nonnormal operators in [10]; also quantitative versions of these results have been obtained in [9] and [11].

The purpose of this paper is to pursue our investigation of the trace vanishing phenomenon of certain trace class commutators. In §2 we improve the previously known results [10, Theorems 3 and 4] about zero-trace commutators involving square roots of normal operators. Zero-trace commutators involving essentially unitary operators and Cesàro operators are discussed in §3.

Let $B(H)$ denote the algebra of all bounded linear operators acting on a separable complex Hilbert space H. Let $C_1(H)$ denote the ideal of trace class operators in $B(H)$. If $T \in C_1(H)$ or $T \in B(H)$ is positive and if $\{f_n\}$ is an orthonormal basis for H, then the trace of T, denoted by $\operatorname{tr} T$ and defined by $\operatorname{tr} T = \sum_n (Tf_n, f_n)$, is independent of the choice of $\{f_n\}$. A compact

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operator $T \in B(H)$ is said to be in the Schatten p-class $C_p(H)$ $(1 \le p \le \infty)$ if $\operatorname{tr}|T|^p < \infty$, where $|T| = (T^*T)^{1/2}$ is the absolute value of T. It is known that $C_p(H)$ $(1 \le p \le \infty)$ becomes a Banach space under the norm $\|T\|_p = (\operatorname{tr}|T|^p)^{1/p}$. Note that $C_2(H)$ is the ideal of Hilbert-Schmidt operators and $C_\infty(H)$ is the ideal of compact operators with $\|\cdot\|_\infty$ denoting the usual operator norm. If X and Y are in $B(H)$ such that both XY and YX lie in $C_1(H)$, then $\operatorname{tr}(XY) = \operatorname{tr}(YX)$, (see [16, Corollary 3.8] or [14, Lemma 2.1]). In particular if $X \in C_p(H)$ and $Y \in C_p(H)$, where $1/p + 1/q = 1$, then XY and YX lie in $C_1(H)$ and $\operatorname{tr}(XY) = \operatorname{tr}(YX)$. If X and Y are in $B(H)$ such that $0 \le X \le Y$ and if $Y \in C_p(H)$, then $X \in C_p(H)$. These definitions and properties can be carried over to the spaces $B(H_1, H_2)$ and $C_p(H_1, H_2)$ of operators from a Hilbert space H_1 into a Hilbert space H_2. We refer the reader to [6, 16] for further properties of the Schatten p-classes.

2. Roots of normal operators

Utilizing a result of Radjavi and Rosenthal [15, Theorem 1] concerning the structure of the square roots of normal operators, the author proved in [10, Theorem 3] that if $T \in B(H)$ with T^2 normal, $T^*T - TT^* \in C_1(H)$, and if $X \in C_2(H)$ such that $TX - XT \in C_1(H)$, then $\operatorname{tr}(TX - XT) = 0$. This generalizes Weiss' result [18, Theorem 8]. Using a structure result due to Gilfeather enables us to prove a general result of this sort that is valid for arbitrary nth roots of normal operators. Moreover if $n = 2$, then we show that the assumption $T^*T - TT^* \in C_1(H)$ can be removed. To achieve our goal we need some preliminary results.

Lemma 1. *If $T \in B(H)$ is nilpotent, i.e., $T^n = 0$ for some integer $n > 1$, and if $T^*T - TT^* \in C_p(H)$ for some p, $1 \le p \le \infty$, then $T \in C_{2p}(H)$.*

Proof. Since $T^n = 0$, it follows by Theorem 1 in [7] that there exists a decomposition $H = \bigoplus_{i=1}^n H_i$ of H into the direct sum of an orthogonal family of subspaces $H_1, H_2, \ldots, H_n$ such that the matrix of T corresponding to this decomposition has the upper triangular form

$$T = \begin{bmatrix} 0 & T_{12} & \cdots & T_{1n} \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & T_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $TT^* - T^*T \in C_p(H)$, it follows that every (i, j) entry of the operator matrix $TT^* - T^*T$ is in $C_p(H_j, H_i)$ $(1 \le i, j \le n)$. The $(1, 1)$ entry of this matrix is $|T_{12}^*|^2 + |T_{13}^*|^2 + \cdots + |T_{1n}^*|^2 \in C_p(H_1)$. Since $|T_{12}^*|^2 + |T_{13}^*|^2 + \cdots + |T_{1n}^*|^2 \ge |T_{1j}^*|^2 \ge 0$ for $2 \le j \le n$, it follows that $|T_{1j}^*|^2 \in C_p(H_1)$ for $2 \le j \le n$. Hence $T_{1j}^* \in C_{2p}(H_1, H_j)$, and so $T_{1j} \in C_{2p}(H_j, H_1)$ for $2 \le j \le n$. The $(2, 2)$ entry of $TT^* - T^*T$ is $|T_{23}^*|^2 + \cdots + |T_{2n}^*|^2 - |T_{12}|^2 \in C_p(H_2)$. Since $|T_{12}|^2 \in C_p(H_2)$, it follows that $|T_{23}^*|^2 + \cdots + |T_{2n}^*|^2 \in C_p(H_2)$. Hence $T_{2j} \in C_{2p}(H_j, H_2)$ for $3 \le j \le n$. By repeating this argument we can show that $T_{ij} \in C_{2p}(H_j, H_i)$ for $1 \le i, j \le n$. Therefore $T \in C_{2p}(H)$, as required.

Lemma 2. *Let $T \in B(H)$ be similar to a normal operator $N \in B(H)$. If $X \in C_2(H)$ and $TX - XT \in C_1(H)$, then* $\operatorname{tr}(TX - XT) = 0$.

Proof. Assume that $T = P^{-1}NP$ for some invertible operator $P \in B(H)$. Now

$$\begin{aligned} TX - XT &= P^{-1}NPX - XP^{-1}NP \\ &= P^{-1}(NPXP^{-1} - PXP^{-1}N)P \\ &= P^{-1}(NY - YN)P, \quad \text{where } Y = PXP^{-1} \in C_2(H). \end{aligned}$$

Therefore $NY - YN \in C_1(H)$, and so by Weiss' result $\operatorname{tr}(NY - YN) = 0$. Because similarity preserves the trace (see [6, Corollary 8.1]), it follows that $\operatorname{tr}(TX - XT) = \operatorname{tr}(P^{-1}(NY - YN)P) = \operatorname{tr}(NY - YN) = 0$.

Lemma 3. *Let $T \in B(H)$ with $T^2 = 0$. If $X \in B(H)$ and $TX - XT \in C_1(H)$, then* $\operatorname{tr}(TX - XT) = 0$.

Proof. Since $T^2 = 0$, it follows that H can be decomposed as $H = H_1 \oplus H_2$ for some orthogonal subspaces H_1, H_2 and with respect to this decomposition $T = \left[\begin{smallmatrix} 0 & T_{12} \\ 0 & 0 \end{smallmatrix}\right]$. Now if $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right]$ is the operator matrix of X corresponding to the above decomposition of H, then

$$TX - XT = \begin{bmatrix} T_{12}X_{21} & T_{12}X_{22} - X_{11}T_{12} \\ 0 & -X_{21}T_{12} \end{bmatrix}.$$

Since $TX - XT \in C_1(H)$, it follows that $T_{12}X_{21} \in C_1(H_1)$ and $X_{21}T_{12} \in C_1(H_1)$, and so by Corollary 3.8 in [16], we have

$$\operatorname{tr}(TX - XT) = \operatorname{tr}(T_{12}X_{21}) - \operatorname{tr}(X_{21}T_{12}) = 0.$$

Now we are in a position to establish the main results of this section.

Theorem 4. *Let $T \in B(H)$ be such that T^n is normal for some integer $n \ge 2$ and $T^*T - TT^* \in C_1(H)$. If $X \in C_2(H)$ and $TX - XT \in C_1(H)$, then* $\operatorname{tr}(TX - XT) = 0$.

Proof. Since T^n is normal, it follows from a result of Gilfeather [5, Theorem 3.1] that there exist reducing subspaces $\{H_i\}_{i=0}^{\infty}$ for T such that $H = \bigoplus_{i=0}^{\infty} H_i$, $T_0 = T|H_0$ is nilpotent with $T_0^n = 0$, and $T_i = T|H_i$ is similar to a normal operator N_i for $i = 1, 2, \ldots$. Let $X = [X_{ij}]$ be the matrix representation of X with respect to the above decomposition of H. Since $T = \bigoplus_{i=0}^{\infty} T_i$, it follows that the (i, i) entry of $TX - XT$ is $T_iX_{ii} - X_{ii}T_i$ for $i = 0, 1, 2, \ldots$. Since $TX - XT \in C_1(H)$ and $X \in C_2(H)$, it follows that $T_iX_{ii} - X_{ii}T_i \in C_1(H_i)$ and $X_{ii} \in C_2(H_i)$ for $i = 0, 1, 2 \ldots$ (see [6, p. 94]). Similarly $T^*T - TT^* \in C_1(H)$ implies that $T_0^*T_0 - T_0T_0^* \in C_1(H_0)$. Hence by Lemma 1, $T_0 \in C_2(H_0)$, and so $\operatorname{tr}(T_0X_{00}) = \operatorname{tr}(X_{00}T_0)$. By Lemma 2, we also have $\operatorname{tr}(T_iX_{ii} - X_{ii}T_i) = 0$

for $i = 1, 2, \ldots$. Therefore $\operatorname{tr}(TX - XT) = \sum_{i=0}^{\infty} \operatorname{tr}(T_i X_{ii} - X_{ii} T_i) = 0$, as desired.

Following the proof of Theorem 4 and using Lemma 3 instead of Lemma 1, we now improve Theorems 3 and 4 in [10] as follows.

Theorem 5. *Let $T \in B(H)$ with T^2 normal. If $X \in C_2(H)$ and $TX - XT \in C_1(H)$, then* $\operatorname{tr}(TX - XT) = 0$.

Proof. With the same notations as in the proof of Theorem 4, we have that $\operatorname{tr}(T_i X_{ii} - X_{ii} T_i) = 0$ for $i = 1, 2, \ldots$. Since $T_0^2 = 0$ and $T_0 X_{00} - X_{00} T_0 \in C_1(H_0)$, it follows by Lemma 3 that $\operatorname{tr}(T_0 X_{00} - X_{00} T_0) = 0$. So $\operatorname{tr}(TX - XT) = \sum_{i=0}^{\infty} \operatorname{tr}(T_i X_{ii} - X_{ii} T_i) = 0$ and the proof is complete.

We conclude this section with the following remarks.

1. If $T^2 = 0$, then it follows from Lemma 1 and the basic properties of the Schatten p-norms that $\|T^*T - TT^*\|_p = 2^{1/p}\|T\|_{2p}^2$ for $1 \le p \le \infty$. Moreover, if $T^n = 0$ for some integer $n > 2$, then estimates relating $\|T^*T - TT^*\|_p$ and $\|T\|_{2p}$ can be obtained using some recent inequalities for the Schatten p-norms of partitioned operator matrices [3].

It should be noted that for $p = \infty$, Lemma 1 remains true under the weaker assumption that T is quasinilpotent. To see this, one needs to formulate the problem in the Calkin algebra setting. However, it is not known to the author whether Lemma 1 remains true for $p \neq \infty$ under the quasinilpotency assumption. A well-known, related result in this direction asserts that if T is quasinilpotent such that $T - T^* \in C_p(H)$ for some p, $1 < p \le \infty$, then $T \in C_p(H)$ (see [6, p. 215]).

2. It follows from the structure theorem of Radjavi and Rosenthal [15, Theorem 1] that if T^2 is normal and if $T^*T - TT^* \in C_p(H)$ for some p, $1 \le p \le \infty$, then $T = N + K$, where N is normal and $K \in C_{2p}(H)$ with $NK = -KN$ (see the proof of Theorem 3 in [10]). Also it follows from Lemma 1 and from a result of Apostol [1] that if T^n and T^m are normal for some relatively prime integers n, m with $m > n \ge 2$ and if $T^*T - TT^* \in C_p(H)$ for some p, $1 \le p \le \infty$, then $T = N + K$, where N is normal, $K \in C_{2p}(H)$, $K^n = 0$, and $NK = KN = 0$.

3. If T^n is normal for some integer $n > 2$ and T is invertible, then by a result of Stampfli [17], T is similar to a normal operator. Thus if $X \in C_2(H)$ and $TX - XT \in C_1(H)$, then $\operatorname{tr}(TX - XT) = 0$ by Lemma 2.

3. Essentially unitary operators and Cesàro operators

Recall that an operator $A \in B(H)$ is said to be essentially unitary if $\pi(A)$ is unitary, where π is the canonical map of $B(H)$ onto the Calkin algebra $B(H)/C_\infty(H)$. Equivalently, A is essentially unitary if both $1 - A^*A$ and $1 - AA^*$ are in $C_\infty(H)$.

If $A \in B(H)$ and $X \in C_\infty(H)$ such that both $AX - XA$ and $A^*X - XA^*$ are in $C_1(H)$, then it follows by the Helton-Howe lemma [8, Lemma 1.3] that

$\operatorname{tr}(AX - XA) = 0$ (see also [19]). Consequently if $A \in B(H)$ is an isometry of finite multiplicity (in particular if A is unitary) and if $X \in C_\infty(H)$ and $AX - XA \in C_1(H)$, then $\operatorname{tr}(AX - XA) = 0$ (see [10, Theorems 2 and 5]).

In the same spirit, we have the following more general result.

Theorem 6. *Let $A \in B(H)$ be such that both $1 - A^*A$ and $1 - AA^*$ are in $C_p(H)$ for some p, $1 \le p \le \infty$. If $X \in C_q(H)$, where $1/p + 1/q = 1$, and if $AX - XA \in C_1(H)$, then $\operatorname{tr}(AX - XA) = 0$.*

Proof. By the Helton-Howe lemma, it is sufficient to show that $A^*X - XA^* \in C_1(H)$. Assume that $A^*A = 1 + K_1$ and $AA^* = 1 + K_2$, where K_1 and K_2 are both in $C_p(H)$. Since $AX - XA \in C_1(H)$, it follows that $A^*(AX - XA)A^* \in C_1(H)$, and so $(1 + K_1)XA^* - A^*X(1 + K_2) \in C_1(H)$. In view of the fact that K_1X and XK_2 are both in $C_1(H)$, it now follows that $A^*X - XA^* \in C_1(H)$, which is the desired conclusion.

For an account on essentially unitary operators, the reader is referred to [13] and references therein.

As an application of Theorem 6 we now discuss commutators involving Cesàro operators.

The discrete Cesàro operator K_0 acting on the Hilbert space l^2 of square summable complex sequences is defined by

$$K_0(a_1, a_2, a_3, \dots) = \left(a_1, \frac{a_1 + a_2}{2}, \frac{a_1 + a_2 + a_3}{3}, \dots\right).$$

With respect to the standard orthonormal basis $\{e_n\}$ for l^2, the matrix of K_0 is given by

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Two continuous analogs of K_0 are the Cesàro operators K_1 and K_∞ defined on the Hilbert spaces $L^2(0, 1)$ and $L^2(0, \infty)$, respectively, by

$$(K_1 f)(x) = \frac{1}{x}\int_0^x f(y)\,dy \quad \text{for } 0 < x < 1$$

and

$$(K_\infty f)(x) = \frac{1}{x}\int_0^x f(y)\,dy \quad \text{for } 0 < x < \infty.$$

The basic properties of these operators were first discussed by Brown, Halmos, and Shields in [4]. It was shown in [4, Theorems 3, 4, and 5] that K_0 is a hyponormal operator, $1 - K_1^*$ is the unilateral shift operator on $L^1(0, 1)$, and $1 - K_\infty^*$ is the bilateral shift operator on $L^2(0, \infty)$. In [12, Theorem 4] it was shown that K_0 is in fact subnormal.

Using the fact that $1-K_1^*$ is the unilateral shift operator on $L^2(0, 1)$, we can easily show that if $X \in C_\infty(L^2(0, 1))$ such that $K_1X - XK_1 \in C_1(L^2(0, 1))$, then $K_1^*X - XK_1^* \in C_1(L^2(0, 1))$ and so $\operatorname{tr}(K_1X - XK_1) = 0$. Note that a similar result for the operator K_∞ can be stated in the obvious way.

It is worth while to remark that as in the continuous cases, the operator $1-K_0$ is essentially unitary. In fact it has been shown in [4] that if $T = 1 - K_0$, then $1 - TT^*$ is the diagonal operator defined by

$$(1 - TT^*)e_n = \tfrac{1}{n}e_n \quad \text{for } n = 1, 2, \ldots.$$

Consequently, $1 - TT^* \in C_p(l^2)$ $(1 < p \leq \infty)$. Since T is a hyponormal contraction (in fact $\|T\| = 1$; see [4]) it follows that $0 \leq 1 - T^*T \leq 1 - TT^*$ and so $1 - T^*T \in C_p(l^2)$ $(1 < p \leq \infty)$. Thus it follows easily from Theorem 6 that if $X \in C_q(l^2)$ for some q, $1 \leq q < \infty$ and if $K_0X - XK_0 \in C_1(l^2)$, then $\operatorname{tr}(K_0X - XK_0) = 0$.

Another relevant property of the Cesàro operator K_0 that deserves attention is that $K_0^*K_0 - K_0K_0^* \in C_1(l^2)$. This can be seen via the Berger-Shaw result [2, Theorem 1], since K_0 is a hyponormal operator with a cyclic vector (see [12, Theorem 1]). More simply, $K_0^*K_0 - K_0K_0^* \in C_1(l^2)$ follows from the fact that $K_0^*K_0 - K_0K_0^*$ is a positive operator with a finite trace. Indeed $\operatorname{tr}(K_0^*K_0 - K_0K_0^*) = 1$. To see this, observe that $K_0e_n = \sum_{j=n}^\infty \frac{1}{j}e_j$ and $K_0^*e_n = \frac{1}{n}\sum_{j=1}^n e_j$ for $n = 1, 2, \ldots$. Thus

$$\begin{aligned}((K_0^*K_0 - K_0K_0^*)e_n, e_n) &= \|K_0e_n\|^2 - \|K_0^*e_n\|^2 \\ &= \left(\sum_{j=n}^\infty \frac{1}{j^2}\right) - \frac{1}{n} \quad \text{for } n = 1, 2, \ldots.\end{aligned}$$

If $s_n = \sum_{m=1}^n((K_0^*K_0 - K_0K_0^*)e_m, e_m)$ for $n = 1, 2, \ldots$, then it is not difficult to see that

$$s_n = \sum_{j=1}^\infty \frac{n}{(j+n)^2} \quad \text{for } n = 1, 2, \ldots.$$

Using the facts that

$$\sum_{j=1}^\infty \frac{1}{(j+n)^2} = \int_0^1 \frac{x^n \log x}{x-1}\,dx,$$

and

$$1 < \frac{\log x}{x-1} < \frac{1}{\sqrt{x}} \quad \text{for } 0 < x < 1,$$

we obtain that

$$n\int_0^1 x^n\,dx < s_n < n\int_0^1 x^{n-1/2}\,dx;$$

hence $\frac{n}{n+1} < s_n < \frac{n}{n+1/2}$ for $n = 1, 2, \ldots$. Consequently, $\operatorname{tr}(K_0^*K_0 - K_0K_0^*) = \lim_{n\to\infty} s_n = 1$.

Finally, we would like to ask the following question concerning the Cesàro operator K_0.

Question. Is it true that $\operatorname{tr}(K_0X - XK_0) = 0$ whenever $X \in C_\infty(l^2)$ and $K_0X - XK_0 \in C_1(l^2)$?

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