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ON INTERSECTION OF COMPACTA IN EUCLIDEAN SPACE II

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ABSTRACT. Suppose that X is a compact subset of n-dimensional Euclidean space $\mathbb{R}^n$. If every map $f: Y \to \mathbb{R}^n$ of a compactum Y can be approximated by a map avoiding X then $\dim X \times Y < n$.

0. INTRODUCTION

The main result of this paper is the following Theorem 1 which is the inverse of the main result of [D1].

Theorem 1. *Let X be a compactum in Euclidean space $\mathbb{R}^n$ and suppose that for a compact metric space Y, the space of maps $C(Y, \mathbb{R}^n - X)$ is dense in $C(Y, \mathbb{R}^n)$. Then* $\dim X \times Y < n$.

Here $C(X, Z)$ denotes the space of continuous maps between X and Z with the compact-open topology.

This theorem plus Theorem 1 from [D1] implies the following:

Theorem 2. *For an arbitrary compactum Y and for a codimension three tame compact subset $X \subset \mathbb{R}^n$ the following are equivalent:*

(a) $C(Y, \mathbb{R}^n - X)$ *is dense in* $C(Y, \mathbb{R}^n)$,
(b) $\dim X \times Y < n$.

Remark 1 [D2]. There exist compacta X, Y with $\dim X = \dim Y = n - 2$ and $\dim X \times Y < n$ for arbitrary n.

The proof of Theorem 1 is based on Spanier-Whitehead duality and some elements of extension theory. The top of the preliminary work in extension

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theory is Theorem 3 which is a generalization of the inequality $\dim \geq \dim_S$, where $\dim_S$ is the stable cohomotopy dimension [D3].

1. Extension theory

Notation [Ku]. Let Y and M be topological spaces; then

$Y\tau M \iff$ for arbitrary closed subset $A \subset Y$ and arbitrary continuous map $\varphi\colon A \to M$ there exists a continuous extension $\overline{\varphi}\colon Y \to M$.

We will consider only the case when M is a CW-complex.

Proposition 1. *Let A be a closed subset of Y. Then the property $Y\tau M$ implies the property $A\tau M$.*

Proposition 2. *Let U be an open subset of a metric space Y. Then $Y\tau M$ implies $U\tau M$.*

Proof. There exists a filtration $F_1 \subset F_2 \subset \cdots \subset F_i \subset \cdots$ of U, where $F_i = Y - O_{1/i}(Y - U)$, where $O_\varepsilon(B)$ denotes the open ε-neighborhood of B in Y. For arbitrary $\varphi\colon A \to M$ by induction construct a sequence of maps $\varphi_i\colon A \cup F_i \to M$ with the property $\varphi_{i+1}|_{A\cup F_i} = \varphi_i$ and $\varphi_i|_A = \varphi$ for all i. The union $\bigcup \varphi_i$ is a continuous extension of the map φ to U.

Proposition 3. *Let B be a closed subset in Y. Then the properties $B\tau M$ and $(Y - B)\tau M$ imply $Y\tau M$.*

Proof. Suppose that $\varphi\colon A \to M$ is an arbitrary map and A is an arbitrary closed subset in Y. Due to the property $B\tau M$ there exists an extension $\varphi'\colon A \cup B \to M$. Since $M \in ANE$ there exists an extension $\overline{\varphi}\colon O \to M$ to an open set $O \supset A \cup B$. Let W be an open set with $A \cup B \subset W \subset [W] \subset O$ where $[W]$ is the closure of W. Apply $Y - B\tau M$ to extend the map $\overline{\varphi}|_{\partial W}$ to the map $\widetilde{\varphi}\colon Y - W \to M$. The union $\widetilde{\varphi} \cup \overline{\varphi}|_{[W]}$ is a continuous map $\psi\colon Y \to M$ with the restriction $\psi|_A = \varphi$.

Proposition 4. *Suppose that compact Y is a union $\bigcup_{i=1}^m F_i$ of closed subsets. Then the properties $F_i\tau M$ imply $Y\tau M$.*

Proof. By induction on m and using Proposition 1.

Let ΣM denote the suspension of M.

Lemma 1. *The property $X\tau M$ implies $(X \times [0, 1])\tau\Sigma M$ for metric spaces X.*

Proof. Define an open set $V(r, U) = \{(x, t) \in X \times [0, 1] | x \in U\}$ and $r - d(x, X - U) < t < r + d(x, X - U)$ where U is an open subset in X and d is the metric on X. Then for every $V(r, U)$ the boundary $\partial V(r, U)$ in $X \times R$ is equal to the union $F_1 \cup F_2$ where

$$F_1 = \{(x, t_1 : t = \max\{0, r - d(x, X - U)\}$$

and

$$F_2 = \{(x, t) : t = \min\{1, r + d(x, X - U)\}.$$

It is easy to see that F_i, $i = 1, 2$ is homeomorphic to the closure $[U]$ of U. since $F_i \tau M$ by Proposition 4 we have $\partial V \tau M$.

Let $A \subset X \times [0, 1]$ be a closed subset and $\varphi: A \to \Sigma M$ be a continuous map. The suspension ΣM consists of the union of two cones: $\Sigma M = \mathrm{con}^+ M \cup \mathrm{con}^- M$ with the vertices x^+ and x^-. Denote $\varphi^{-1}(x^+) = A^+$ and $\varphi^{-1}(x^-) = A^-$. Since the sets $V(r, U)$ generate a basis of the topology on $X \times [0, 1]$ there exists an open set V such that $A^+ \subset V$, $A^- \subset X \times [0, 1] - [V]$ and $V = \bigcup_{i=1}^m V(r_i, u_i)$. Since $\partial V \subset \bigcup_{i=1}^m \partial V(r_i, u_i)$, Proposition 1 and 4 imply that $\partial V \tau M$. Since $\Sigma M - \{x^+, x^-\}$ is homeomorphic to $M \times \mathbb{R}$ there exists an extension $\varphi': \partial V \to \Sigma M - \{x^+, x^-\}$ of the map $\varphi|_{A \cap \partial V}$. Let ψ be the union of φ' and φ. Consider the restrictions $\psi^+ = \psi|_{(A \cap [V]) \cup \partial V}$ and $\psi^- = \psi|_{(A-V) \cup \partial V}$. Since spaces $\mathrm{con}^+ M$ and $\mathrm{con}^- M$ are contractable there are extensions $\overline{\psi}^+: [V] \to \mathrm{con}^+ M$ and $\overline{\psi}^-: X \times [0, 1] - V \to \mathrm{con}^- M$ of ψ^+ and ψ^-. The union $\overline{\psi}^+ \cup \overline{\psi}^-$ gives the map $\overline{\psi}: X \times [0, 1] \to \Sigma M$ which is an extension of φ.

Proposition 5. *The property $X \tau M$ implies* $\mathrm{con}\, X \tau \Sigma M$.

Proof. By virtue of Lemma 1, $X \times [0, 1] \tau \Sigma M$. By Proposition 2 we have $X \times [0, 1) \tau \Sigma M$. Since $pt \tau \Sigma M$ and $\mathrm{con}\, X - pt \approx X \times [0, 1)$ Proposition 3 implies $\mathrm{con}\, X \tau \Sigma M$.

The following proposition has a similar proof.

Proposition 6. *The property $X \tau M$ implies $\Sigma X \tau \Sigma M$.*

Corollary. *$\forall i\, X \tau M$ implies $\Sigma^i X \tau \Sigma^i M$.*

Theorem 3. *For arbitrary CW-complex M and compactum X the property $X \tau M$ implies the property $X \tau \Omega^i \Sigma^i M$ for any $i = 1, 2, \dots, \infty$.*

Here $\Omega^i Y$ denotes iterated loop space.

Proof. Let $i < \infty$. Consider the diagram

$$\begin{array}{ccc} [\Sigma^i A, \Sigma^i M] & \longrightarrow & [A, \Omega^i \Sigma^i M] \\ \uparrow & & \uparrow \\ [\Sigma^i X, \Sigma^i M] & \longrightarrow & [X, \Omega^i \Sigma^i M]. \end{array}$$

All horizontal arrows are isomorphisms, the left vertical arrow is an epimorphism due to the corollary of Proposition 6. Thus the right vertical arrow is an epimorphism too. Hence $X \tau \Omega^i \Sigma^i M$.

Recall that $\Omega^\infty \Sigma^\infty M = \varinjlim \Omega^i \Sigma^i M$. Then compactness of X and the properties $X \tau \Omega^i \Sigma^i M$ for $i < \infty$ imply the property $X \tau \Omega^\infty \Sigma^\infty M$.

Remark 2. $X \tau \Omega^\infty \Sigma^\infty S^n \iff \dim_S X \le n$ where $\dim_S$ is the stable cohomotopy dimension [D3]. So, for $M = S^n$, Theorem 3 claims the inequality $\dim X \ge \dim_S X$.

Problem. Does $X\tau M$ imply $X\tau SP^{\infty}M$, where SP^{∞} is the infinite symmetric power?

2. Spanier-Whitehead duality

Lemma 2. *Let U be an open n-dimensional ball and $X \subset U$ be a closed subset. Let $M = U - X$, and let X' be the one-point compactification of X. Then for a finite-dimensional compactum Y and for large enough m there is an isomorphism $\beta_Y\colon [Y, \Omega^m\Sigma^m M] \to [\Sigma^m(Y \wedge X'), S^{m+n-1}]$ which depends naturally on Y.*

A proof of Lemma 2 actually is contained in [D4] (see the lemma) and it is a consequence of Spanier-Whitehead duality.

Consider the one-point compactification U' of U. In the n-dimensional sphere $U' \simeq S^n$ choose a decreasing sequence $\{K_i\}$ of polyhedra with intersection $\bigcap K_i = X'$. Spanier-Whitehead duality claims that for finite-dimensional Y and large enough m there is an isomorphism $\beta_i\colon [\Sigma^m Y, \Sigma^m(U' - K_i)] \to [\Sigma^m(Y \wedge K_i), S^{m+n-1}]$ [Sp]. The limit $\lim_{i\to\infty}\beta_i$ gives the isomorphism β_Y. This conclusion is based on the following propositions.

Proposition 7. *Let the compactum X be a limit space of an inverse system $\{X_i, p_i^{i+1}\}$ of compacta and let M be a CW-complex. Then $[X, M] = \varinjlim [X_i, M]$.*

Proposition 8. *Suppose M is a limit space of a direct system $\{M_i, \varphi_i^{i+1}\}$ of CW-complexes and inclusions. Then for any compactum Y there is an equality $[Y, M] = \varinjlim [Y, M_i]$.*

Lemma 3. *Let X be a compact subset of $\mathbb{R}^n$, then for compactum Y the following are equivalent*:

(1) *the space $C(Y, \mathbb{R}^n - X)$ is dense in $C(Y, \mathbb{R}^n)$,*
(2) *for any open ball $U \subset \mathbb{R}^n$, $Y\tau M_U$, where $M_U = U - X$.*

Proof. (1) $\Rightarrow$ (2). Let A be a closed subset of Y and $\varphi\colon A \to M_U$ be an arbitrary map. Choose an arbitrary extension $\varphi'\colon Y \to U$. There exists $\varepsilon > 0$ such that every map ψ ε-close to φ is homotopic to ψ in M_U. Approximate φ' by $\psi\colon Y \to \mathbb{R}^n - X$ such that $\psi(Y) \subset M_U$ and ψ is ε-close to φ'. The homotopy extension theorem implies that there is an extension $\bar{\varphi}\colon Y \to M_U$ of φ.

(2) $\Rightarrow$ (1) See the proof of Theorem 1 in [D1].

3. Proof of Theorem 1

Suppose the contrary: $\dim X \times Y \geq n$. We can assume that $\dim X \times Y = n$. (Otherwise consider $X = X \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$ where $m = \dim X \times Y$. It is easy to see that $C(Y, \mathbb{R}^m - X)$ is dense in $C(Y, \mathbb{R}^m)$. The case of $\dim Y =$

∞ is excluded because if $\dim Y \geq n$ then always there is an essential map of Y onto n-dimensional cube.)

Let Z be a cone over Y. Lemma 3 implies the property $Y\tau M_U$ for an arbitrary open ball $U \subset \mathbb{R}^n$. By virtue of Proposition 5 we have $Z\tau\Sigma M_U$ for every open ball $U \subset \mathbb{R}^n$. Denote $W = U \cap X$ and let W' be the one-point compactification of W. Let V be an arbitrary open subset of Z and $A = Z - V$.

The set U is naturally embedded in ΣU as the equator. Then $\Sigma U - W$ is homotopy equivalent to ΣM_U. By Lemma 2 there is the following diagram

$$\begin{array}{ccc} [A, \Omega^k\Sigma^k\Sigma M_U] & \xrightarrow{\beta_1} & [W' \wedge A, \Omega^k\Sigma^k S^n] \\ \uparrow \alpha_1 & & \uparrow \alpha_2 \\ [Z, \Omega^k\Sigma^k\Sigma M_U] & \xrightarrow{\gamma_1} & [W' \wedge Z, \Omega^k\Sigma^k S^n], \end{array}$$

where for large enough k all horizontal arrows are isomorphisms. Theorem 3 implies that α_1 is an epimorphism. Therefore α_2 is an epimorphism.

Since $\dim(Z \wedge W') = n + 1 < 2n - 1$ the homomorphisms β_2, γ_2 in the following diagram are isomorphisms:

$$\begin{array}{ccc} [W' \wedge A, \Omega^k\Sigma^k S^n] & \xleftarrow{\beta_2} & [W' \wedge A, S^n] \\ \uparrow \alpha_2 & & \uparrow \alpha_3 \\ [W' \wedge Z, \Omega^k\Sigma^k S^n] & \xleftarrow{\gamma_2} & [W' \wedge Z, S^n]. \end{array}$$

Hence α_3 is an epimorphism.

Since $\dim W' \times Z = n + 1$ then for homomorphism β_3, γ_3 in the diagram

$$\begin{array}{ccc} [W' \wedge A, S^n] & \xrightarrow{\beta_3} & [W' \wedge A, K(\mathbb{Z}, n)] \\ \uparrow \alpha_3 & & \uparrow \alpha_4 \\ [W' \wedge Z, S^n] & \xrightarrow{\gamma_3} & [W' \wedge Z, K(\mathbb{Z}, n)], \end{array}$$

are epimorphisms. Therefore α_4 is an epimorphism. Here $K(\mathbb{Z}, n)$ is the Eilenberg-MacLane complex.

Consider the cohomology exact sequence of the pair $(W' \wedge Z, W' \wedge A)$: $\cdots \leftarrow \check{H}^{n+1}(W' \wedge Z) \leftarrow \check{H}^{n+1}(W' \wedge Z, W' \wedge A) \leftarrow \check{H}^n(W' \wedge A) \xleftarrow{\alpha_4} \check{H}^n(W' \wedge Z)$. Since Z is contractible then $\check{H}^*(W' \wedge Z) = 0$. This and the fact that α_4 is an epimorphism imply that $\check{H}^{n+1}(W' \wedge Z, W' \wedge A) = 0$. Because $(W' \wedge Z)/(W' \wedge A) = (W' \times Z)/(W' \times A) \cup (*) \times Z$ (where $(*) = W' - W$) then $\check{H}^{n+1}(W' \times Z, W'_U \times A \cup (*) \times Z) = 0$. In other terms $H_c^{n+1}(W \times V) = 0$. Since the sets of the form $W \times V$ are a basis in $X \times Z$ we have [Kuz] the inequality $c - \dim_{\mathbb{Z}}(X \times Z) < n + 1$. Since both X and Y are finite-dimensional then $\dim(X \times \operatorname{con} Y) < n + 1$. We have reached a contradiction.

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