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ON THE RELATIVE STRENGTH OF TWO ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. In this paper we prove a theorem concerning the relative strength of $|R, p_n|_k$ and $|R, q_n|_k$ summability methods, $k > 1$, that generalizes a result of Bosanquet [1].

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By u_n we denote the nth $(C, 1)$ mean of the sequence (s_n). The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). We say that the series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.4}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|R, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|R, p_n|$) summability.

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A summability method P is said to be stronger than another summability method Q, if the summability of a series by the method Q implies its summability by the method P. If, in addition, the method P sums the series to the same sum as that obtained by Q, the method F is said to include the method Q.

2. Bosanquet's theorem

The following result concerning the relative strength of two absolute summability methods is due to Bosanquet [1].

Theorem A. *Suppose that* $p_n > 0$, $P_n \to \infty$ *and suppose similarly that* $q_n > 0$, $Q_n \to \infty$. *In the case* $k = 1$, *in order that*

$$|R, p_n|_k \Rightarrow |R, q_n|_k \tag{2.1}$$

it is necessary and sufficient that

$$q_n P_n / p_n Q_n = O(1). \tag{2.2}$$

3. Main theorem

We may ask whether a like result holds for $k > 1$. Therefore, the aim of this paper is to prove a theorem concerning the relative strength of $|R, p_n|_k$ and $|R, q_n|_k$ summability methods, with $k > 1$. Now, we shall prove the following:

Theorem. *Let* $k > 1$. *In order for* (2.1) *to hold,* (2.2) *is necessary. If we suppose that*

$$\sum_{n=v}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left\{ \frac{v^{k-1} q_v^{k-1}}{Q_v^k} \right\}, \tag{3.1}$$

then (2.2) *is also sufficient.*

Remark. If we take $k = 1$, then condition (3.1) is obvious.

4. Lemma

We need the following lemma for the proof of our theorem.

Lemma. *Let* $k \geq 1$ *and let* $A = (a_{nv})$ *be an infinite matrix. In order that* $A \in (l^k; l^k)$ *it is necessary that*

$$a_{nv} = O(1) \qquad (\textit{all } n, v). \tag{4.1}$$

Proof. Since $l \subset l^k \subset l^\infty$ it follows that if $A \in (l^k; l^k)$, then we must have

$$A \in (l; l^\infty). \tag{4.2}$$

However, (4.1) is necessary for (4.2). This completes the proof of the lemma.

5. Proof of the theorem

Necessity. Now, let the (R, p_n) transform of $\sum a_n$, expressed in series form, be $\sum b_n$, and let the (R, q_n) transform of $\sum a_n$, expressed in series form, be

$\sum c_n$. Then, for $n \geq 1$, we have

$$b_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v. \tag{5.1}$$

Hence

$$P_{n-1} a_n = \frac{P_{n-1} P_n}{p_n} b_n - \frac{P_{n-1} P_{n-2}}{p_{n-2}} b_{n-1}.$$

And therefore,

$$a_n = \frac{P_n}{p_n} b_n - \frac{P_{n-2}}{p_{n-1}} b_{n-1}. \tag{5.2}$$

(In the case $n = 1$, we take $P_{-1} = 0$.) Also, for $n \geq 1$, we have

$$c_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v. \tag{5.3}$$

If we put (5.2) in (5.3), we get

$$\begin{aligned} c_n &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} \left(\frac{P_v}{p_v} b_v - \frac{P_{v-2}}{p_{v-1}} b_{v-1} \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{b_v}{p_v} \left(Q_{v-1} P_v - P_{v-1} Q_v \right) + \frac{q_n P_n}{p_n Q_n} b_n. \end{aligned} \tag{5.4}$$

Now, from (4.1) we can write down the matrix transforming $(n^{1-1/k} b_n)$ into $(n^{1-1/k} c_n)$. The assertion (2.1) is equivalent to the assertion that this matrix $\in (l^k; l^k)$. Hence, by the lemma, a necessary condition for (2.1) is that the elements of this matrix should be bounded. In particular, it is necessary that the diagonal elements should be bounded, and this gives us the result that (2.2) is necessary.

Sufficiency. Let $c_{n,1}$ denote the sum on the right-hand side of (5.4) and let $c_{n,2}$ denote the second term on the right-hand side of (5.4). Suppose the conditions are satisfied. Then, it is enough to show that, if

$$\sum_{n=1}^{\infty} n^{k-1} |b_n|^k < \infty, \tag{5.5}$$

then

$$\sum_{n=1}^{\infty} n^{k-1} |c_{n,i}|^k < \infty \qquad (i = 1, 2). \tag{5.6}$$

For $i = 2$ this is an immediate corollary of (2.2). To complete the proof of the sufficiency, it is enough to show that

$$\sum_{n=1}^{\infty} n^{k-1} |c_{n,1}|^k < \infty.$$

We have

$$Q_{v-1}P_v - P_{v-1}Q_v = -P_v q_v + p_v Q_v = O(p_v Q_v), \quad \text{by (2.2)}.$$

Now, applying Hölder's inequality with $k > 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1}|c_{n,1}|^k &= O(1)\sum_{n=1}^{\infty} n^{k-1}\frac{q_n^k}{(Q_n Q_{n-1})^k}\left\{\sum_{v=1}^{n-1}|b_v|Q_v\right\}^k \\
&= O(1)\sum_{n=1}^{\infty}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}}\left\{\sum_{v=1}^{n-1}|b_v|^k q_v (Q_v/q_v)^k\right\} \times \left\{\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1} q_v\right\}^{k-1} \\
&= O(1)\sum_{n=1}^{\infty}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}}\left\{\sum_{v=1}^{n-1}|b_v|^k q_v (Q_v/q_v)^k\right\} \\
&= O(1)\sum_{v=1}^{\infty}|b_v|^k q_v (Q_v/q_v)^k \sum_{n=v}^{\infty}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}} \\
&= O(1)\sum_{v=1}^{\infty}|b_v|^k q_v (Q_v/q_v)^k \frac{v^{k-1}q_v^{k-1}}{Q_v^k}, \quad \text{by (3.1)}.
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} n^{k-1}|c_{n,1}|^k = O(1)\sum_{v=1}^{\infty} v^{k-1}|b_v|^k < \infty, \quad \text{by (5.5)}.$$

This completes the proof of the theorem.

If we take $p_n = 1$ for all values of n, then we get the following corollary from our theorem.

Corollary. *Suppose that* $q_n > 0$, $Q_n \to \infty$. *Let* $k > 1$. *In order that* $|C, 1|_k \Rightarrow |R, q_n|_k$ *it is necessary that*

$$(i) \qquad nq_n = O(Q_n).$$

If condition (3.1) *is satisfied, then condition* (i) *is also sufficient.*

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References

1. L. S. Bosanquet, MR **11** (1950), 654.
2. T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113–141.
3. G. H. Hardy, *Divergent series*, Oxford Univ. Press, Oxford, 1949.

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