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DISCRETIZATION IN THE METHOD OF AVERAGING

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ABSTRACT. Let $f: R \times R^{\overline{m}} \times R \to R^{\overline{m}}$, $f = f(\varepsilon, x, t)$ be a C^2-mapping 1-periodic in t having the form $f(0, x, t) = Ax + o(|x|)$ as $x \to 0$ where $A \in \mathscr{L}(R^{\overline{m}})$ has no eigenvalues with zero real parts. We study the relation between local stable manifolds of the equation

$$x' = \varepsilon \cdot f(\varepsilon, x, t), \qquad \varepsilon > 0 \text{ is small}$$

and of its discretization

$$x_{n+1} = x_n + (\varepsilon/m) \cdot f(\varepsilon, x_n, t_n),$$
$$t_{n+1} = t_n + 1/m,$$

where $m \in \{1, 2, \dots\} = \mathscr{N}$. We show behavior of these manifolds of the discretization for the following cases: (a) $m \to \infty$, $\varepsilon \to \overline{\varepsilon} > 0$, (b) $m \to \infty$, $\varepsilon \to 0$, (c) $m \to k \in \mathscr{N}$, $\varepsilon \to 0$.

1. INTRODUCTION

Let us consider the equation

$$x' = \varepsilon \cdot f(\varepsilon, x, t), \tag{1.1}$$

where $f \in C^2(R \times R^{\overline{m}} \times R,\ R^{\overline{m}})$, f is 1-periodic in t, and $\varepsilon \in R$ is a small parameter. We assume that f has the form

$$f(0, x, t) = Ax + g(x, t)$$

with hyperbolic $A \in \mathscr{L}(R^{\overline{m}})$ i.e., A has no eigenvalues with zero real parts and $g(x, t) = o(|x|)$ as $x \to 0$. It is well known [5] that (1.1) has a 1-periodic solution $\overline{z}(\varepsilon, \cdot)$ for each small $\varepsilon \neq 0$ such that $\overline{z}(\varepsilon, \cdot) \to 0$ as $\varepsilon \to 0$. The equation (1.1) has the discretization for each $m \in \mathscr{N} \backslash \{1\} = \mathscr{N}_1$

$$\begin{aligned} x_{n+1} &= x_n + (\varepsilon/m) \cdot f(\varepsilon, x_n, t_n), \\ t_{n+1} &= t_n + 1/m. \end{aligned} \tag{1.2}$$

The purpose of this paper is to study the relation between (1.1) and its discretization (1.2). First we shall show (§2) the existence of an invariant curve of (1.2) for small ε which tends to $\overline{z}(\varepsilon, \cdot)$ as $m \to \infty$. Then in §3 we investigate

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behavior of local stable manifolds of these invariant curves. We show that they tend to the local stable manifold of $\overline{z}(\varepsilon, \cdot)$ as $m \to \infty$ for $\varepsilon > 0$. The main result of this paper is Theorem 3.6 where all possible cases regarding limit behavior of these manifolds are shown. Similar problems have been studied in [3, 4, 6]. Our method is directly related to [3].

2. Invariant curves

In this section we find invariant curves of (1.2) for ε small and $m \in \mathcal{N}$, $m \geq 2$. Let us consider the equation

$$\begin{aligned} x_2 &= x_1 + \frac{\varepsilon}{m} \cdot f(\varepsilon, x_1, t) \\ x_3 &= x_2 + \frac{\varepsilon}{m} \cdot f\left(\varepsilon, x_2, t + \frac{1}{m}\right) \\ &\vdots \\ x_m &= x_{m-1} + \frac{\varepsilon}{m} \cdot f\left(\varepsilon, x_{m-1}, t + \frac{m-2}{m}\right) \\ x_1 &= x_m + \frac{\varepsilon}{m} \cdot f\left(\varepsilon, x_m, t + \frac{m-1}{m}\right). \end{aligned} \tag{2.1}$$

If $\overline{x} = (x_1, \dots, x_m)$, $A_m\overline{x} = (x_2 - x_1, \dots, x_1 - x_m)$, and

$$F_m(\varepsilon, \overline{x}, t) = \left(f(\varepsilon, x_1, t), \dots, f\left(\varepsilon, x_m, t + \frac{m-1}{m}\right)\right)$$

then (2.1) has the form

$$A_m\overline{x} = \frac{\varepsilon}{m} \cdot F_m(\varepsilon, \overline{x}, t).$$

Theorem 2.1. *There is $\varepsilon_0 > 0$ and a C^1-mapping*

$$z : \mathcal{N}_1 \times \langle -\varepsilon_0, \varepsilon_0 \rangle \times R \to R^{\overline{m}}$$

satisfying

(i) $z = z(m, \varepsilon, t)$ *is* 1*-periodic in* t, $z(\cdot, 0, \cdot) = 0$;
(ii) *the sequence* $\{z(m, \varepsilon, t + (n/m)), t + (n/m)\}_{-\infty}^{\infty}$ *satisfies* (2.1) *i.e., the set* $\{(z(m, \varepsilon, t), t)\}_{t \in R}$ *is invariant for the discretization* (1.2);
(iii) $\lim_{m \to \infty} z(m, \varepsilon, \cdot) = \overline{z}(\varepsilon, \cdot)$ *in the space* $C^1(\langle 0, 1\rangle, R^{\overline{m}})$.

Proof. We follow [2]. It is clear that $\operatorname{Ker} A_m = \{(x_1, \dots, x_m), x_1 = \cdots = x_m\}$ and we define $P_m : (R^{\overline{m}})^m \to (R^{\overline{m}})^m$ by

$$P_m\overline{x} = \left(\frac{x_1 + \cdots + x_m}{m}, \dots, \frac{x_1 + \cdots + x_m}{m}\right).$$

P_m is a projection and our equation has the form:

$$A_m\overline{x} = (\varepsilon/m)(I - P_m)F_m(\varepsilon, \overline{x}, t), \tag{2.2}$$

$$0 = P_m F_m(\varepsilon, \overline{x}, t), \tag{2.3}$$

since $(R^{\overline{m}})^m = \operatorname{Ker} A_m \oplus \operatorname{Im} A_m$. By [2] we know that $|(A_m/\operatorname{Im} A_m)^{-1}| \le c \cdot m$, hence (2.2) has the form

$$\overline{x}_2 = (\varepsilon/m) B_m^{-1}(I - P_m) F_m(\varepsilon, \overline{x}_1 + \overline{x}_2, t) \tag{2.4}$$

and $(1/m) \cdot |B_m^{-1}| \le c$, where $\overline{x}_1 \in \operatorname{Ker} A_m$, $\overline{x}_2 \in \operatorname{Im} A_m$, $B_m = A_m/\operatorname{Im} A_m$. We notice that $\operatorname{Ker} A_m = \operatorname{Im} P_m$ and $\operatorname{Im} A_m = \operatorname{Ker} P_m$. Further

$$\begin{aligned} |D_x F_m(\varepsilon, \overline{x}, t)v| &= \sqrt{|D_x f\left(\varepsilon, x_1, t)v_1\right|^2 + \cdots + |D_x f(\varepsilon, x_m, t + \frac{m-1}{m}\Big) v_m|^2} \\ &\le M \cdot |v|, \quad \text{where } M = \max_{\varepsilon \in \langle -1, 1\rangle,\ |w| \le 1,\ t \in \langle 0, 1\rangle} |D_x f(\varepsilon, w, t)|. \end{aligned}$$

By using the implicit function theorem these facts imply that (2.4) has a unique small solution $\overline{x}_2(t, m, \overline{x}_1, \varepsilon)$ for each small ε, $m \in \mathcal{N}_1$, $t \in \langle 0, 1\rangle$, $\overline{x}_1$ bounded. Hence we obtain the bifurcation equation from (2.3):

$$0 = P_m F_m(\varepsilon, \overline{x}_1 + \overline{x}_2(t, m, \overline{x}_1, \varepsilon), t),$$

i.e.,

$$0 = \frac{1}{m} \sum_{i=1}^{m} f(\varepsilon, z + x_2^i(t, m, \overline{x}_1, \varepsilon), t), \tag{2.5}$$

where $\overline{x}_1 = (z, \ldots, z) \in \operatorname{Ker} A_m$, $z \in R^{\overline{m}}$, $\overline{x}_2 = (x_2^1, \ldots, x_2^m)$. Since $\overline{x}_2(t, m, \overline{x}_1, 0) = 0$, $f(0, 0, t) = 0$, and (2.5) has the form $Az + (o(z) + 0(\varepsilon)) = 0$ as $z \to 0$, $\varepsilon \to 0$, we obtain a solution of (2.5) again by the implicit function theorem and thus we have a solution $(x_1^m(t, \varepsilon), \ldots, x_m^m(t, \varepsilon))$ of (2.1) uniformly for each ε small, $m \in \mathcal{N}$, $m \ge 2$, and $t \in \langle 0, 1\rangle$.

Using the periodicity of f we obtain

$$x_i^m(t, \varepsilon) = x_1^m\left(t + \frac{i-1}{m}, \varepsilon\right), \qquad x_1^m(t+1, \varepsilon) = x_1^m(t, \varepsilon).$$

In the same way as we solved (2.1) we can see that $x_1^m(\cdot, \cdot)$, $\frac{\partial}{\partial t} x_1^m(\cdot, \cdot)$, $\frac{\partial^2}{\partial t^2} x_1^m(\cdot, \cdot)$ are bounded on $\langle -\varepsilon_0, \varepsilon_0\rangle \times \langle 0, 1\rangle$ uniformly for $m \ge 2$.

Let us choose an arbitrary sequence $\{(m_i, \varepsilon_i)\}_0^\infty$ such that $m_i \to \infty$, $\varepsilon_i \to \varepsilon \in \langle -\varepsilon_0, \varepsilon_0\rangle$, $\varepsilon \ne 0$. Then by (2.1)

$$x_1^{m_i}\left(t + \frac{1}{m_i}, \varepsilon_i\right) = x_1^{m_i}(t, \varepsilon_i) + \frac{\varepsilon_i}{m_i} \cdot f(\varepsilon_i, x_1^{m_i}(t, \varepsilon_i), t).$$

Hence for some $d_j^i \in \langle 0, 1\rangle$ by the mean-value theorem

$$\frac{\partial}{\partial t} x_{1j}^{m_i}\left(t + \frac{d_j^i}{m_i}, \varepsilon_i\right) \cdot \frac{1}{m_i} = \frac{\varepsilon_i}{m_i} \cdot \overline{f}_j(\varepsilon_i, x_1^{m_i}(t, \varepsilon_i), t), \tag{$-$}$$

where $x_1^{m_i} = (x_{11}^{m_i}, \ldots, x_{1\overline{m}}^{m_i})$, $f = (\overline{f}_1, \ldots, \overline{f}_{\overline{m}})$. Since $x_1^{\cdot}(\cdot, \cdot)$, $\frac{\partial}{\partial t} x_1^{\cdot}(\cdot, \cdot)$, and $\frac{\partial^2}{\partial t^2} x_1^{\cdot}(\cdot, \cdot)$ are bounded on $\mathcal{N}_1 \times \langle -\varepsilon_0, \varepsilon_0\rangle \times \langle 0, 1\rangle$, by the Arzela-Ascoli theorem $\{x_1^{m_i}(\cdot, \varepsilon_i)\}_0^\infty$ has a convergent subsequence that tends to $z(\cdot)$ in $C^1(\langle 0, 1\rangle, R^{\overline{m}})$ and moreover, we have from $(-)$

$$z' = \varepsilon \cdot f(\varepsilon, z, t), \qquad \varepsilon \neq 0.$$

Utilizing the fact that (1.1) has a unique small 1-periodic solution for $\varepsilon \neq 0$ we obtain the proof of Theorem 2.1. We have used the following evident argument: Let X, Y be topological spaces and let $f\colon D \to Y$ be a mapping defined on $\varnothing \neq D \subset X$. If for some $x \in X$, $y \in Y$ and each sequence $\{x_i\}_0^\infty \subset D$ such that $x_i \to x$ as $i \to \infty$, the sequence $\{f(x_i)\}_0^\infty$ has a subsequence tending to y, then $\lim_{z\to x} f(z) = y$.

3. Invariant manifolds

We investigate local stable manifolds of (1.1) and (1.2). Let $A = \operatorname{diag}(B, C)$, where $B \in \mathscr{L}(R^{m_1})$, $C \in \mathscr{L}(R^{m_2})$ have positive, negative real parts of their eigenvalues respectively. We note that (1.1) has the averaged equation

$$(+) \qquad x' = \int_0^1 f(0, x, t)\, dt.$$

Theorem 3.1. *There is $\delta > 0$ and a C^1-mapping*

$$h\colon \mathscr{N}_1 \times (0, \delta) \times B_\delta = \{v \in R^{m_2}, |v| \leq \delta\} \times R \to R^{m_1}$$

such that

(i) *$h = h(m, \varepsilon, v, t)$ is 1-periodic in t;*
(ii) *the graph of $h(m, \varepsilon, \cdot, \cdot)$ in $R^{\overline{m}} \times R$ is a local stable manifold of $\{(z(m, \varepsilon, t), t)\}_{t\in R}$ for (1.2);*
(iii) *let graph $z_{1,\varepsilon}$, graph z_2 be local stable manifolds of $\{(\overline{z}(\varepsilon, t), t)\}_{t\in R}$ for (1.1) and $0 \in R^{\overline{m}}$ for $(+)$, respectively. Then*
 (a) $\lim_{m\to\infty, \varepsilon\to\overline{\varepsilon}\in(0,\delta)} h(m, \varepsilon, \cdot, \cdot) = z_{1,\overline{\varepsilon}}$,
 (b) $\lim_{m\to\infty, \varepsilon\to 0} h(m, \varepsilon, \cdot, \cdot) = \overline{z}_2$

in the space $C_p^1(B_\delta \times R, R^{m_1}) = \{h \in C^1(B_\delta \times R, R^{m_1}),\ h = h(v, t)$ is 1-periodic in $t\}$. Here $\overline{z}_2(v, t) = z_2(v)$, $z_2 \in C^2(B_\delta, R^{m_1})$, $z_{1,\varepsilon} \in C_p^2(B_\delta \times R, R^{m_1})$.

Proof. (1.2) has the form

$$(3.1) \qquad \begin{aligned} u_{n+1} &= u_n + \tfrac{\varepsilon}{m} B u_n + \tfrac{\varepsilon}{m} g_1(\varepsilon, u_n + v_n, t_n) = u_n + \tfrac{\varepsilon}{m} f_1(\varepsilon, u_n + v_n, t_n), \\ v_{n+1} &= v_n + \tfrac{\varepsilon}{m} C v_n + \tfrac{\varepsilon}{m} g_2(\varepsilon, u_n + v_n, t_n) = v_n + \tfrac{\varepsilon}{m} f_2(\varepsilon, u_n + v_n, t_n), \\ t_{n+1} &= t_n + \tfrac{1}{m}. \end{aligned}$$

Similarly as in [3] we can show the existence of a mapping $h(m, \varepsilon)\colon B_{\overline{\delta}} \times R \to R^{m_1}$ for some $\overline{\delta} > 0$, uniformly for $\varepsilon > 0$ small, $m \in \mathscr{N}_1$ such that

(i) $h(m, \varepsilon)(\cdot, \cdot)$ is 1-periodic in $t \in R$;
(ii) the graph of $h(m, \varepsilon)$ in $R^{\overline{m}} \times R$ is a local stable manifold of the set $\{(z(m, \varepsilon, t), t)\}_{t\in R}$ for (3.1);

(iii) $D_v^i D_t^j h(m, \varepsilon)(\cdot, \cdot)$ is uniformly bounded on $B_{\overline{\delta}} \times \langle 0, 1\rangle$ for $(m, \varepsilon) \in \mathcal{N}_1 \times (0, \varepsilon_0\rangle$ for $i + j \leq 2$.

Since the graph of $h(m, \varepsilon)$ is locally invariant for (1.2), $h = h(m, \varepsilon)$ satisfies on $B_\delta \times R$ for some δ, $0 < \delta < \overline{\delta}$

$$(3.2)\quad h(v + \tfrac{\varepsilon}{m} f_2(\varepsilon, h(v, t) + v, t), t + \tfrac{1}{m}) = h(v, t) + \tfrac{\varepsilon}{m} f_1(\varepsilon, h(v, t) + v, t).$$

Let us choose a sequence $m_i \to \infty$, $\varepsilon_i \to \varepsilon \in (0, \delta\rangle$. Then by the Arzela–Ascoli theorem $\{h(m_i, \varepsilon_i)\}_0^\infty$ has a subsequence which tends to z in $C_p^1(B_\delta \times R, R^{m_1})$. On the other hand, it follows from (3.2) by the mean-value theorem for some s_j^i, $r_j^i \in \langle 0, 1\rangle$

(3.3)

$$\begin{aligned}
&h_j^i\left(v + \frac{\varepsilon_i}{m_i} f_2(\varepsilon_i, h^i(v, t) + v, t), t + \frac{1}{m_i}\right) - h_j^i(v, t)\\
&\quad = h_j^i\left(v, t + \frac{1}{m_i}\right) - h_j^i(v, t) + h_j^i\left(v + \frac{\varepsilon_i}{m_i} f_2(\varepsilon_i, h^i(v, t) + v, t), t + \frac{1}{m_i}\right)\\
&\quad\quad - h_j^i\left(v, t + \frac{1}{m_i}\right)\\
&\quad = D_v h_j^i\left(v + s_j^i \frac{\varepsilon_i}{m_i} f_2(\varepsilon_i, h^i(v, t) + v, t), t + \frac{1}{m_i}\right)\\
&\quad\quad \cdot \frac{\varepsilon_i}{m_i} f_2(\varepsilon_i, h^i(v, t) + v, t) + D_t h_j^i\left(v, t + \frac{r_j^i}{m_i}\right) \cdot \frac{1}{m_i}\\
&\quad = \frac{\varepsilon_i}{m_i} \cdot f_1^j(\varepsilon_i, h^i(v, t) + v, t),
\end{aligned}$$

where $h^i = h(m_i, \varepsilon_i)$, $h^i = (h_1^i, \ldots, h_{m_1}^i)$, $f_1 = (f_1^1, \ldots, f_1^{m_1})$. Since $\varepsilon_i \to \varepsilon$, $m_i \to \infty$ we have

$$(3.4)\qquad \varepsilon \cdot D_v z(v, t) \cdot f_2(\varepsilon, z + v, t) + D_t z(v, t) = \varepsilon \cdot f_1(\varepsilon, z + v, t).$$

But this equation has a unique solution near $0 \times R \subset R^{\overline{m}} \times R$ whose graph in $R^{\overline{m}} \times R$ is precisely the local stable manifold of $\{(\overline{z}(\varepsilon, t), t)\}_{t \in R}$. This follows in the same way as in [3] from the two facts: first, a solution of this equation is a locally invariant set of (1.1) and second, only the local stable manifold of $\overline{z}(\varepsilon, \cdot)$ is a graph of such a mapping near $0 \times R \subset R^{\overline{m}} \times R$.

Now we study a similar case when $m_i \to \infty$, $\varepsilon_i \to 0$. Then we modify (3.2) and (3.3) in the following way using the mean-value theorem and the fact

$$\sum_{i=1}^{m} h\left(v, t + \frac{i}{m}\right) - h\left(v, t + \frac{i-1}{m}\right) = 0$$

$$\sum_{i=1}^{m_p} D_v h_j^p \left(v + \frac{\varepsilon_p}{m_p} \cdot v_j^p, t + \frac{i}{m_p}\right) \cdot \frac{\varepsilon_p}{m_p}$$

$$(3.5) \qquad \cdot f_2\left(\varepsilon_p, h^p\left(v, t + \frac{i-1}{m_p}\right) + v, t + \frac{i-1}{m_p}\right)$$

$$= \frac{\varepsilon_p}{m_p} \sum_{i=1}^{m_p} f_1^j \left(\varepsilon_p, h_j^p(\left(v, t + \frac{i-1}{m_p}\right) + v, t + \frac{i-1}{m_p}\right).$$

Now we apply the following well-known result

Lemma 3.2. *Let $f: R \to R^{\overline{n}}$ be a 1-periodic C^1-mapping. Then for each $t \in R$*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} f\left(t + \frac{i-1}{m}\right) = \int_0^1 f(s)\, ds.$$

Hence from (3.5) we have

$$\int_0^1 D_v z_j(v, t) \cdot f_2(0, z + v, t)\, dt = \int_0^1 f_1^j(0, z + v, t)\, dt.$$

Note that we assume that $\{h(m_i, \varepsilon_i)\}_0^\infty$ has a subsequence which tends to z in $C_p^1(B_\delta \times R, R^{m_1})$. On the other hand, it follows from (3.3) that z is independent of t, since $D_t z = 0$. Thus $z = z(v)$ and

$$(3.6) \qquad Dz(v) \int_0^1 f_2(0, z(v) + v, t)\, dt = \int_0^1 f_1(0, z(v) + v, t)\, dt.$$

We see that this equation and (3.4) are similar. Indeed, (3.6) is the equation of the local manifold of $0 \in R^{\overline{m}}$ for the averaged equation of (1.1)

$$x' = \int_0^1 f(0, x, t)\, dt.$$

This completes the proof of Theorem 3.1.

Remark 3.3. By the paper [3] we know that

$$\lim_{\varepsilon\to 0} z_{1,\varepsilon} = \overline{z}_2$$

in the space $C_p^1(B_\delta \times R, R^{m_1})$.

A case remains: $\varepsilon_i \to 0$, $m_i = m$. If $\{h(m, \varepsilon_i)(\cdot, \cdot)\}_0^\infty$ tends to z in $C_p^1(B_\delta \times R, R^{m_1})$ then by (3.2)

$$z(v, t + (1/m)) = z(v, t),$$

i.e., z is $(1/m)$-periodic in t. It follows from (3.5) that

$$D_v z(v, t) \cdot \frac{1}{m} \sum_{i=1}^{m} f_2\left(0, z(v, t) + v, t + \frac{i-1}{m}\right)$$

$$= \frac{1}{m} \sum_{i=1}^{m} f_1\left(0, z + v, t + \frac{i-1}{m}\right),$$

i.e., the graph of $z/B_\delta \times \{t\}$ is a local stable manifold of $0 \in R^{\overline{m}}$ of the following equation, where now $t \in R$ is a parameter

$$x' = \frac{1}{m}\sum_{i=1}^{m} f\left(0, x, t + \frac{i-1}{m}\right). \tag{3.7}$$

Hence we obtain

Theorem 3.4. $\{h(m_i, \varepsilon_i)\}_0^\infty$ *tends in* $C_p^1(B_\delta \times R, R^{m_1})$ *to the family of local stable manifolds of* $0 \in R^{\overline{m}}$ *for* (3.7) *depending on* t, *as* $m_i \to m$, $\varepsilon_i \to 0$.

Let the family of local stable manifolds of $0 \in R^{\overline{m}}$ for (3.7) depending on t is a graph of w_m for some $w_m \in C_p^2(B_\delta \times R, R^{m_1})$. We know that

$$\frac{1}{m}\sum_{i=1}^{m} f\left(0, x, t + \frac{i-1}{m}\right) = Ax + \frac{1}{m}\sum_{i=1}^{m} g\left(x, t + \frac{i-1}{m}\right).$$

From this it follows that we can derive w_m in a standard way [1, 3] uniformly for $m \in \mathscr{N}_1$. Moreover $D_v^i D_t^j w_m(\cdot, \cdot)$ are also uniformly bounded on $B_\delta \times \langle 0, 1\rangle$ for $i + j \le 2$.

Utilizing this fact and Lemma 3.2 we obtain by the previous method

Theorem 3.5. $\lim_{m\to\infty} w_m = \overline{z}_2$ (*see Theorem* 3.1) *in* $C_p^1(B_\delta \times R, R^{m_1})$.

Lastly, let $\mathscr{N}^* = \mathscr{N}_1 \cup \{\infty\}$ be a compactification of $\mathscr{N}_1$. Now we define a mapping $H: \mathscr{N}^* \times \langle 0, \delta\rangle \to C_p^1(B_\delta \times R, R^{m_1})$ for $\delta > 0$ sufficiently small in the following way

$$H(\cdot, \cdot) = \begin{cases} h(m, \varepsilon, \cdot, \cdot) & \text{for } (m, \varepsilon) \in \mathscr{N}_1 \times (0, \delta\rangle, \\ w_m & \text{for } (m, 0),\ m \in \mathscr{N}_1, \\ z_{1,\varepsilon} & \text{for } (\infty, \varepsilon),\ \varepsilon > 0, \\ \overline{z}_2 & \text{for } (\infty, 0). \end{cases}$$

Summarizing all the previous results we have the main theorem of this paper:

Theorem 3.6. *The above defined mapping* $H: \mathscr{N}^* \times \langle 0, \delta\rangle \to C_p^1(B_\delta \times R, R^{m_1})$ *is continuous.*

4. Appendix

To help the reader understand this paper we prove the result from [2] that was used in the proof of Theorem 2.1; namely, that $|(A_m/\operatorname{Im} A_m)^{-1}| \le c \cdot m$ for some constant c and each $m \in \mathscr{N}_1$. We know that $A_m: (R^{\overline{m}})^m \to (R^{\overline{m}})^m$,

$$A_m x = (x_2 - x_1, \dots, x_1 - x_m), \qquad x = (x_1, \dots, x_m),\ x_i \in R^{\overline{m}}.$$

We have the scalar product on $(R^{\overline{m}})^m$

$$\langle x, y\rangle = \sum_{i=1}^{m} (x_i, y_i),$$

where $(\cdot, \cdot)$ is the standard scalar product on $R^{\overline{m}}$. On the other hand, $\langle A_m x, A_m x\rangle = \langle -B_m x, x\rangle$ where

$$B_m x = (x_2 + x_m - 2x_1, \dots, x_{i+1} + x_{i-1} - 2x_i, \dots, x_1 + x_{m-1} - 2x_m).$$

Lemma 4.1. *The spectrum of* B_m *is* $\{-4\sin^2 \frac{\pi}{m} \cdot j,\ j = 0, \dots, [m/2]\}$.

Proof. It is clear that $B_m x = bx$ if and only if $x_1, x_2, \dots, x_m$ is a periodic orbit of the dynamical system

$$x_{i+2} = (b+2)x_{i+1} - x_i, \qquad b \le 0. \tag{4.1}$$

Without loss of generality we can assume that $\overline{m} = 1$. The equation (4.1) has a general solution in the form

(i) $C_1 r_1^j + C_2 r_2^j$, $r_{1,2}^2 - (b+2)r_{1,2} + 1 = 0$, $\overline{C}_1 = C_2$, C_i are complex numbers, $b \ne 0, 4$

(ii) $C_1 j + C_2$ for $b = 0$, $(-1)^j \cdot (C_1 j + C_2)$ for $b = -4$, C_i are real constants.

Hence $0 \in \sigma(B_m)$ and $-4 \in \sigma(B_m)$ for m even. The case (i) is more difficult. If $C_1 r_1^j + C_2 r_2^j$ is an m-periodic orbit of (4.1) then

$$C_1 + C_2 = C_1 r_1^m + C_2 r_2^m, \qquad \overline{C}_1 = C_2 \ne 0$$
$$C_1 r_1 + C_2 r_2 = C_1 r_1^{m+1} + C_2 r_2^{m+1}.$$

Hence

$$\det \begin{pmatrix} r_1^m - 1, & r_2^m - 1 \\ r_1^{m+1} - r_1, & r_2^{m+1} - r_2 \end{pmatrix} = 0.$$

Thus

$$(r_1^m - 1) \cdot (r_2^m - 1) \cdot (r_2 - r_1) = 0.$$

Since $r_1 \ne r_2$, $r_1 \cdot r_2 = 1$, we obtain $r_1^m = r_2^m = 1$ and

$$r_1 = \cos \frac{2\pi}{m} j + \underline{i} \cdot \sin \frac{2\pi}{m} j, \qquad j = 0, \dots, m-1,$$

where $\underline{i}^2 = -1$. Finally, by (i)

$$\begin{aligned} b &= \frac{r_1^2 + 1}{r_1} - 2 \\ &= \left(1 + \cos 2\frac{2\pi}{m} j + \underline{i} \cdot \sin 2\frac{2\pi}{m} j\right) \\ &\quad \cdot \left(\cos \frac{2\pi}{m} j - \underline{i} \cdot \sin \frac{2\pi}{m} j\right) - 2 \\ &= -4 \cdot \sin^2 \frac{\pi}{m} j. \end{aligned}$$

It follows from Lemma 4.1 that

$$|(A_m / \operatorname{Im} A_m)^{-1} \le 1 / \left(2 \cdot \sin \frac{\pi}{m}\right) \le c \cdot m$$

for some constant $c > 0$.

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