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ON ISOMORPHISMS OF INDUCTIVE LIMIT C^*-ALGEBRAS

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ABSTRACT. We prove that for a large class of inductive limit C^*-algebras, including inductive limits of finite direct sums of interval and circle algebras, any $*$-isomorphism is induced from an approximate intertwining, in the sense of Elliott, between the inductive systems defining the algebras.

In [3] Elliott introduced a notion, called an approximate intertwining, between two sequences of C^*-algebras, and used it successfully to extend, beyond the AF-algebras, the class of C^*-algebras for which K-theory is a complete invariant. The purpose of this note is to show that for a considerable class of inductive limit C^*-algebras, including the inductive limits of finite direct sums of interval algebras, $C[0, 1] \otimes M_n$, and circle algebras, $C(\mathbb{T}) \otimes M_n$, any $*$-isomorphism is induced by an approximate intertwining. So with this notion Elliott has grasped all isomorphisms of such C^*-algebras. This result shows to what extent the inductive limit C^*-algebra reflects the inductive system defining it and gives a useful tool for the study of the structure of such inductive limit C^*-algebras.

Fix two sequences

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} A_4 \to \cdots \tag{A}$$

and

$$B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} B_4 \to \cdots \tag{B}$$

of C^*-algebras and $*$-homomorphisms. And fix subsets $F_i \subseteq A_i$ and $G_i \subseteq B_i$, which generate A_i and B_i, respectively, as C^*-algebras, $i \in \mathbb{N}$. For $i > j$ we set $\phi_{i,j} = \phi_{i-1} \circ \phi_{i-2} \circ \cdots \circ \phi_j$ and $\psi_{i,j} = \psi_{i-1} \circ \psi_{i-2} \circ \cdots \circ \psi_j$. We define $\phi_{i,i}$ and $\psi_{i,i}$ to be the identity on A_i and B_i, respectively. Let $A = \varinjlim A_i$ and $B = \varinjlim B_i$ denote the corresponding inductive limit C^*-algebras and $\mu_i^A : A_i \to A$ and $\mu_i^B : B_i \to B$ be the canonical $*$-homomorphisms. We emphasize that the connecting $*$-homomorphisms are not assumed to be injective.

The following is one of several possible elaborations on Remark 2.3 of [3].

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Lemma 1 (Elliott). *Let* $\{n(i)\}$ *and* $\{m(i)\}$ *be strictly increasing sequences in* $\mathbb{N}$ *and* $\{\delta_n\}$ *a sequence in* $[0, \infty[$ *such that* $\sum_{n=1}^{\infty} \delta_n < \infty$. *Let* $\alpha_i \colon A_{n(i)} \to B_{m(i)}$, $i \in \mathbb{N}$, *be* $*$*-homomorphisms such that*

$$\|\psi_{m(i+1),m(i)} \circ \alpha_i \circ \phi_{n(i),n(k)}(x) - \alpha_{i+1} \circ \phi_{n(i+1),n(k)}(x)\| < \delta_i,$$

whenever $k \leq i$ *and* $x \in F_{n(k)}$.

Then the sequence $\{\mu^B_{m(i)} \circ \alpha_i \circ \phi_{n(i),k}(x)\}$, $i \geq k$, *converges in* B *for each* $x \in A_k$ *and all* $k \in \mathbb{N}$. *Furthermore, there is a* $*$*-homomorphism* $\alpha \colon A \to B$ *such that*

$$\alpha(\mu^A_k(x)) = \lim_{i\to\infty} \mu^B_{m(i)} \circ \alpha_i \circ \phi_{n(i),k}(x),$$

$x \in A_k$, $k = 1, 2, 3, \ldots$.

Proof. To prove that $\{\mu^B_{m(i)} \circ \alpha_i \circ \phi_{n(i),n(k)}(y)\}$ converges for all $y \in A_{n(k)}$, it clearly suffices to consider the case that $y \in F_{n(k)}$. But in this case, the sequence is Cauchy by our assumption. It follows that the sequence $\{\mu^B_{m(i)} \circ \alpha_i \circ \phi_{n(i),k}(x)\} = \{\mu^B_{m(i)} \circ \alpha_i \circ \phi_{n(i),n(k)} \circ \phi_{n(k),k}(x)\}$ converges for all $x \in A_k$.

Therefore we can define $\alpha'_k \colon A_k \to B$ by $\alpha'_k(x) = \lim_{i\to\infty} \mu^B_{m(i)} \circ \alpha_i \circ \phi_{n(i),k}(x)$. Since $\alpha'_{k+1} \circ \phi_k = \alpha'_k$, we obtain a $*$-homomorphism $\alpha \colon A \to B$ with the stated property. □

The assumption of the lemma can be visualized schematically as follows.

$$\begin{array}{ccccccccc}
A_{n(1)} & \longrightarrow & A_{n(2)} & \longrightarrow & A_{n(3)} & \longrightarrow & A_{n(4)} & \longrightarrow & \cdots \\
{\scriptstyle\alpha_1}\big\downarrow & \delta_1 & {\scriptstyle\alpha_2}\big\downarrow & \delta_2 & {\scriptstyle\alpha_3}\big\downarrow & \delta_3 & {\scriptstyle\alpha_4}\big\downarrow & & \\
B_{m(1)} & \longrightarrow & B_{m(2)} & \longrightarrow & B_{m(3)} & \longrightarrow & B_{m(4)} & \longrightarrow & \cdots
\end{array}$$

A $*$-homomorphism $\alpha \colon A \to B$ obtained from $*$-homomorphisms α_i as in Lemma 1 is called *approximately filtered* and denoted by $\alpha = \varinjlim \alpha_i$.

Definition 2. By an *approximate intertwining* between the sequences (A) and (B), we mean two increasing sequences $\{n(i)\}$ and $\{m(i)\}$ in $\mathbb{N}$, together with $*$-homomorphisms $\alpha_i \colon A_{n(i)} \to B_{m(i)}$ and $\beta_i \colon B_{m(i)} \to A_{n(i+1)}$ such that

$$\|\alpha_{i+1} \circ \beta_i(x) - \psi_{m(i+1),m(i)}(x)\| < 2^{-i},$$

$$\text{for } x \in \bigcup_{k\leq i} \psi_{m(i),m(k)}(G_{m(k)}),\quad x \in \bigcup_{k\leq i} \alpha_i \circ \phi_{n(i),n(k)}(F_{n(k)}),$$

$$\|\beta_i \circ \alpha_i(y) - \phi_{n(i+1),n(i)}(y)\| < 2^{-i},$$

$$\text{for } y \in \bigcup_{k\leq i} \phi_{n(i),n(k)}(F_{n(k)}),\quad y \in \bigcup_{k<i} \beta_{i-1} \circ \psi_{m(i-1),m(k)}(G_{m(k)}),\quad i \in \mathbb{N}.$$

This definition should be compared with that in [3]. Schematically, an approximate intertwining can be visualized as follows.

$$\begin{tikzcd}
A_{n(1)} \arrow[r] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "2^{-1}", pos=0.3] & A_{n(2)} \arrow[r] \arrow[d, "\alpha_2"'] \arrow[dr, phantom, "2^{-2}", pos=0.3] & A_{n(3)} \arrow[r] \arrow[d, "\alpha_3"'] \arrow[dr, phantom, "2^{-3}", pos=0.3] & A_{n(4)} \arrow[r] \arrow[d, "\alpha_4"'] & \cdots \\
B_{m(1)} \arrow[r] \arrow[ur, "\beta_1"] \arrow[ur, phantom, "2^{-1}"', pos=0.7] & B_{m(2)} \arrow[r] \arrow[ur, "\beta_2"] \arrow[ur, phantom, "2^{-2}"', pos=0.7] & B_{m(3)} \arrow[r] \arrow[ur, "\beta_3"] \arrow[ur, phantom, "2^{-3}"', pos=0.7] & B_{m(4)} \arrow[r] & \cdots
\end{tikzcd}$$

This diagram is analogous to the diagram in [1, p. 206], which has played such an important role in the study of AF-algebras. The difference is that the triangles do not commute exactly, but only better and better as one approaches infinity in the diagram.

Theorem 3 (Elliott). *An approximate intertwining between the diagrams* (A) *and* (B) *induces a $*$-isomorphism between $A = \varinjlim A_i$ and $B = \varinjlim B_i$.*

Proof. From the norm estimates in the definition of an approximate intertwining, it follows that

$$\|\psi_{m(k),m(i)} \circ \alpha_i(x) - \alpha_k \circ \phi_{n(k),n(i)}(x)\| \le 2^{-i+2}, \qquad x \in \bigcup_{j \le i} \phi_{n(i),n(j)}(F_{n(j)}),$$

and

$$\|\phi_{n(k),n(i+1)} \circ \beta_i(x) - \beta_{k-1} \circ \psi_{m(k-1),m(i)}(x)\| \le 2^{-i+1},$$
$$x \in \bigcup_{j \le i} \psi_{m(i),m(j)}(G_{m(j)}).$$

Therefore Lemma 1 can be applied to get approximately filtered $*$-homomorphisms $\alpha = \varinjlim \alpha_i \colon A \to B$ and $\beta = \varinjlim \beta_i \colon B \to A$. By using the original estimates from the definition of an approximate intertwining it is easily seen that α and β are inverses of each other. □

Lemma 4. *Assume that the generating sets $F_i \subseteq A_i$ and $G_i \subseteq B_i$ are all finite sets. Let $\alpha \colon A \to B$ be a $*$-isomorphism such that α and α^{-1} both are approximately filtered.*

Then α is induced from an approximate intertwining of the sequences (A) *and* (B).

Proof. Assume that α and $\beta = \alpha^{-1}$ are derived from the data indicated by the following two diagrams:

$$\begin{tikzcd}
A_{n(1)} \arrow[r] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\delta_1"] & A_{n(2)} \arrow[r] \arrow[d, "\alpha_2"'] \arrow[dr, phantom, "\delta_2"] & A_{n(3)} \arrow[r] \arrow[d, "\alpha_3"'] \arrow[dr, phantom, "\delta_3"] & A_{n(4)} \arrow[r] \arrow[d, "\alpha_4"'] & \cdots \\
B_{m(1)} \arrow[r] & B_{m(2)} \arrow[r] & B_{m(3)} \arrow[r] & B_{m(4)} \arrow[r] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
B_{k(1)} \arrow[r] \arrow[d, "\beta_1"'] \arrow[dr, phantom, "\varepsilon_1"] & B_{k(2)} \arrow[r] \arrow[d, "\beta_2"'] \arrow[dr, phantom, "\varepsilon_2"] & B_{k(3)} \arrow[r] \arrow[d, "\beta_3"'] \arrow[dr, phantom, "\varepsilon_3"] & B_{k(4)} \arrow[r] \arrow[d, "\beta_4"'] & \cdots \\
A_{l(1)} \arrow[r] & A_{l(2)} \arrow[r] & A_{l(3)} \arrow[r] & A_{l(4)} \arrow[r] & \cdots .
\end{tikzcd}$$

Let $t_1, t_2, t_3, \dots$ be the sequence in $]0, 1[$ determined recursively as follows:

$$t_1 = 1/6 \quad \text{and} \quad 2t_{n+1} + t_n = 2^{-(n+1)}, \qquad n \geq 1.$$

We construct an approximate intertwining between (A) and (B), which can be described schematically as follows:

$$\begin{tikzcd}
A_{n(y_1)} \arrow[r, "2^{-2}"] \arrow[d] & A_{n(y_2)} \arrow[r, "2^{-3}"] \arrow[d] & A_{n(y_3)} \arrow[r, "2^{-4}"] \arrow[d] & A_{n(y_4)} \arrow[r] \arrow[d] & \cdots \\
B_{k(x_1)} \arrow[r, "2^{-2}"'] \arrow[ur] & B_{k(x_2)} \arrow[r, "2^{-3}"'] \arrow[ur] & B_{k(x_3)} \arrow[r, "2^{-4}"'] \arrow[ur] & B_{k(x_4)} \arrow[r] & \cdots ,
\end{tikzcd}$$

where the $*$-homomorphisms $\alpha_i' \colon A_{n(y_i)} \to B_{k(x_i)}$ are related to the α_i's by

$$\alpha_i' = \psi_{k(x_i), m(c_i)} \circ \alpha_{c_i} \circ \phi_{n(c_i), n(y_i)}, \qquad i \in \mathbb{N}, \tag{1}$$

for some strictly increasing sequence $\{c_i\} \subseteq \mathbb{N}$ with $n(y_i) < n(c_i)$ and $m(c_i) < k(x_i)$. Once this is done it follows from Lemma 1 that $\alpha = \varinjlim \alpha_i' = \varinjlim \alpha_i$, so that α is the $*$-isomorphism induced by the constructed approximate intertwining.

The construction proceeds by induction. To construct α_i', set $y_1 = 1$ and choose any $c_i \in \mathbb{N}$ such that $n(1) < n(c_1)$ and

$$\|\alpha(\mu^A_{n(1)}(x)) - \mu^B_{m(c_1)} \circ \alpha_{c_1} \circ \phi_{n(c_1), n(1)}(x)\| < t_2,$$

for $x \in F_{n(1)}$. This can be done by the definition of $\alpha = \varinjlim \alpha_i$, since $F_{n(1)}$ is a finite set. Then we choose $x_1 \in \mathbb{N}$ such that $m(c_1) < k(x_1)$ and set

$$\alpha_1' = \psi_{k(x_1), m(c_1)} \circ \alpha_{c_1} \circ \phi_{n(c_1), n(1)}.$$

Now assume that we have constructed the following piece of an approximate intertwining:

$$\begin{tikzcd}
A_{n(y_1)} \arrow[r, "2^{-2}"] \arrow[d, "\alpha_1'"'] & A_{n(y_2)} \arrow[r, "2^{-3}"] \arrow[d, "\alpha_2'"'] & \cdots \arrow[r] & A_{n(y_d)} \arrow[d, "\alpha_d'"'] \\
B_{k(x_1)} \arrow[r, "2^{-2}"'] \arrow[ur, "\beta_1'"] & B_{k(x_2)} \arrow[r, "2^{-3}"'] \arrow[ur] & \cdots \arrow[r] & B_{k(x_d)}
\end{tikzcd}$$

such that α_i', $i \leq d$, have the form indicated by (1) and such that

$$\|\alpha(\mu^A_{n(y_d)}(z)) - \mu^B_{m(c_d)} \circ \alpha_{c_d} \circ \phi_{n(c_d), n(y_d)}(z)\| < t_{2d}, \tag{2}$$

for $z \in M_d$, where M_d is the union of

$$\bigcup_{a \leq d} \phi_{n(y_d), n(y_a)}(F_{n(y_a)})$$

and

$$\bigcup_{a<d} \beta'_{d-1} \circ \psi_{k(x_{d-1}), k(x_a)}(G_{k(x_a)}).$$

The induction step then proceeds as follows. Find b in $\mathbb{N}$ such that $l(b) > n(y_d)$, $b > x_d$ and

$$\|\beta(\mu^B_{k(x_d)}(y)) - \mu^A_{l(b)} \circ \beta_b \circ \psi_{k(b), k(x_d)}(y)\| < t_{2d+1}, \tag{3}$$

for $y \in N_d$, where N_d is the union of $\bigcup_{a \leq d} \psi_{k(x_d), k(x_a)}(G_{k(a)})$ and $\alpha'_d(M_d)$. This can be done by the definition of $\beta = \varinjlim \beta_i$, using that N_d is a finite set. Next find y_{d+1} in $\mathbb{N}$ so large that

$$\|\phi_{n(y_{d+1}), l(b)}(z)\| < t_{2d+1} + \|\mu^A_{l(b)}(z)\|, \tag{4}$$

for all $z \in (\beta_b \circ \psi_{k(b), k(x_d)} \circ \alpha'_d - \phi_{l(b), n(y_d)})(M_d)$. This can be done because $\|\mu^A_{l(b)}(z)\| = \lim_{i \to \infty} \|\phi_{n(i), l(b)}(z)\|$ for each z.

Set $\beta'_d = \phi_{n(y_{d+1}), l(b)} \circ \beta_b \circ \psi_{k(b), k(x_d)}$. For each $x \in M_d$, we find the following estimates, using first (4), then (3) and (1), and finally (2):

$$\begin{aligned}
&\|\beta'_d \circ \alpha'_d(x) - \phi_{n(y_{d+1}), n(y_d)}(x)\| \\
&\quad < t_{2d+1} + \|\mu^A_{l(b)} \circ \beta_b \circ \psi_{k(b), k(x_d)} \circ \alpha'_d(x) - \mu^A_{l(b)} \circ \phi_{l(b), n(y_d)}(x)\| \\
&\quad \leq 2t_{2d+1} + \|\beta \circ \mu^B_{k(x_d)} \circ \alpha'_d(x) - \mu^A_{n(y_d)}(x)\| \\
&\quad = 2t_{2d+1} + \|\beta(\mu^B_{m(c_d)} \circ \alpha_{c_d} \circ \phi_{n(c_d), n(y_d)}(x)) - \mu^A_{n(y_d)}(x)\| \\
&\quad \leq 2t_{2d+1} + t_{2d} + \|\beta \circ \alpha \circ \mu^A_{n(y_d)}(x) - \mu^A_{n(y_d)}(x)\| = 2t_{2d+1} + t_{2d} < 2^{-(d+1)}.
\end{aligned}$$

Now choose $c_{d+1} > c_d$ such that $n(c_{d+1}) > n(y_{d+1})$ and

$$\|\mu^B_{m(c_{d+1})} \circ \alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})}(x) - \alpha(\mu^A_{n(y_{d+1})}(x))\| < t_{2d+2}, \tag{5}$$

for all $x \in M_{d+1} \cup \beta'_d(N_d)$ and then x_{d+1} so that $k(x_{d+1}) > m(c_{d+1})$ and

$$\|\psi_{k(x_{d+1}), m(c_{d+1})}(z)\| < t_{2d+2} + \|\mu^B_{m(c_{d+1})}(z)\|, \tag{6}$$

for all $z \in (\alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})} \circ \beta'_d - \psi_{m(c_{d+1}), k(x_d)})(N_d)$.

Set $\alpha'_{d+1} = \psi_{k(x_{d+1}), m(c_{d+1})} \circ \alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})}$. Then we find the following estimates for all $x \in N_d$, by using first (6), then (5) and (3):

$$\begin{aligned}
&\|\alpha'_{d+1} \circ \beta'_d(x) - \psi_{k(x_{d+1}), k(x_d)}(x)\| \\
&\quad < t_{2d+2} + \|\mu^B_{m(c_{d+1})} \circ \alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})} \circ \beta'_d(x) - \mu^B_{k(x_d)}(x)\| \\
&\quad < 2t_{2d+2} + \|\alpha \circ \mu^A_{n(y_{d+1})} \circ \beta'_d(x) - \mu^B_{k(x_d)}(x)\| \\
&\quad < 2t_{2d+2} + t_{2d+1} < 2^{-(d+1)}.
\end{aligned}$$

This completes the induction step and hence the proof. □

Lemma 5. *Let $\alpha\colon A \to B$ be a $*$-homomorphism and assume that the generating subsets $F_i \subseteq A_i$ are finite. Assume, furthermore, that for each $i \in \mathbb{N}$ and each $\varepsilon > 0$, there is a $k \in \mathbb{N}$ and a $*$-homomorphism $\alpha_i\colon A_i \to B_k$ such that*

$$\|\mu_k^B \circ \alpha_i(x) - \alpha \circ \mu_i^A(x)\| < \varepsilon, \qquad x \in F_i.$$

Then α is approximately filtered.

Proof. For each $i \in \mathbb{N}$, the set $F_i' = \bigcup_{j \le i} \phi_{i,j}(F_j)$ is finite. Our assumptions are therefore strong enough to give us a strictly increasing sequence $\{k(i)\}$ in $\mathbb{N}$ and $*$-homomorphisms $\alpha_i\colon A_i \to B_{k(i)}$ such that

$$\|\mu_{k(i)}^B \circ \alpha_i(x) - \alpha \circ \mu_i^A(x)\| < 2^{-i}, \qquad x \in F_i'. \tag{7}$$

We use this to construct by induction a strictly increasing sequence $\{m(i)\}$ in $\mathbb{N}$ and $*$-homomorphisms $\alpha_i'\colon A_i \to B_{m(i)}$ such that $m(i) > k(i)$, $\alpha_i' = \psi_{m(i),k(i)} \circ \alpha_i$, and

$$\|\psi_{m(i+1),m(i)} \circ \alpha_i'(x) - \alpha_{i+1}' \circ \phi_i(x)\| < 2^{-i} + 2^{-i-1}, \qquad x \in F_i'.$$

We leave the reader to start the induction and concentrate on the induction step. So assume that $m(i)$ and α_i', $i \le d$, have been found. (7) yields the estimate

$$\begin{aligned}
&\|\mu_l^B(\psi_{l,m(d)} \circ \alpha_d'(x) - \psi_{l,k(d+1)} \circ \alpha_{d+1} \circ \phi_d(x))\| \\
&\quad = \|\mu_{k(d)}^B \circ \alpha_d(x) - \mu_{k(d+1)}^B \circ \alpha_{d+1} \circ \phi_d(x)\| \\
&\quad < 2^{-d} + 2^{-d-1}, \qquad x \in F_d',
\end{aligned}$$

for $l = \max\{m(d), k(d+1)\}$. It follows that there is a $m(d+1) > l$ such that

$$\|\psi_{m(d+1),m(d)} \circ \alpha_d'(x) - \psi_{m(d+1),k(d+1)} \circ \alpha_{d+1} \circ \phi_d(x)\| < 2^{-d} + 2^{-d-1},$$

$x \in F_d'$. Set $\alpha_{d+1}' = \psi_{m(d+1),k(d+1)} \circ \alpha_{d+1}$. This completes the induction step.

By Lemma 1 we obtain a filtered $*$-homomorphism $\alpha' = \varinjlim \alpha_i'$. Since

$$\begin{aligned}
\alpha'(\mu_k^A(x)) &= \lim_{i\to\infty} \mu_{m(i)}^B \circ \alpha_i' \circ \phi_{i,k}(x) \\
&= \lim_{i\to\infty} \mu_{m(i)}^B \circ \psi_{m(i),k(i)} \circ \alpha_i \circ \phi_{i,k}(x) \\
&= \lim_{i\to\infty} \mu_{k(i)}^B \circ \alpha_i \circ \phi_{i,k}(x)
\end{aligned}$$

and

$$\|\mu_{k(i)}^B \circ \alpha_i \circ \phi_{i,k}(x) - \alpha \circ \mu_i^A \circ \phi_{i,k}(x)\| < 2^{-i},$$

$x \in F_k$, $i \ge k$, we conclude that $\alpha'(\mu_k^A(x)) = \alpha \circ \mu_k^A(x)$, $x \in F_k$, $k \in \mathbb{N}$. Hence $\alpha = \alpha'$ and the proof is complete. □

Let $\mathscr{C}$ denote the class of C^*-algebras A that meet the following conditions:

(i) A is finitely generated, and

(ii) when $B = \lim_n B_n$ is the inductive limit of a sequence of C^*-algebras B_n, $\mu_n\colon B_n \to B$ the canonical $*$-homomorphisms, $\alpha\colon A \to B$ a $*$-homomorphism, $F \subseteq A$ a finite set and $\varepsilon > 0$, then there is a $k \in \mathbb{N}$ and a $*$-homomorphism $\beta\colon A \to B_k$ such that $\|\mu_k \circ \beta(x) - \alpha(x)\| < \varepsilon$, $x \in F$.

If all the C^*-algebras in the sequence (A) are in the class $\mathscr{C}$, the second assumption of Lemma 5 is automatically fulfilled. So we get the following result.

Theorem 6. (i) *Assume that the C^*-algebras occurring in the sequence* (A) *are in $\mathscr{C}$. Then every $*$-homomorphism $\alpha\colon A \to B$ is approximately filtered.*

(ii) *Assume that the C^*-algebras occurring in the sequences* (A) *and* (B) *are in $\mathscr{C}$. Then any $*$-isomorphism $\alpha\colon A \to B$ is induced from an approximate intertwining of the sequences* (A) *and* (B).

Proof. Combine Lemma 4 and Lemma 5. □

By modifying arguments from [2], it is not difficult to prove the following assertions:

(a) $\mathbb{C}$, $C_0(\mathbb{R})$, $C(\mathbb{T})$, $O_A \in \mathscr{C}$ (where O_A are the Cuntz-Krieger algebras).

(b) When A, $B \in \mathscr{C}$ are unital, then $A \oplus B \in \mathscr{C}$.

(c) When $A \in \mathscr{C}$ is unital and F is finite-dimensional, then $A \otimes F \in \mathscr{C}$.

(d) When A, $B \in \mathscr{C}$ are unital and $F \subseteq A$, $F \subseteq B$ is a common unital finite-dimensional C^*-subalgebra, then (the amalgamated free product) $A *_F B \in \mathscr{C}$.

Furthermore, it is asy to show that $\mathscr{C}$ contains $C[0, 1]$ and $C^*(F_n)$ for all n. In particular, $\mathscr{C}$ contains all finite direct sums of circle and interval algebras.

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