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ORTHOGONAL POLYNOMIALS WITH RATIO ASYMPTOTICS

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ABSTRACT. A general construction is given for measures for which the corresponding orthogonal polynomials have ratio asymptotics.

Let ν be a positive Borel measure on the unit circle T and form the orthonormal polynomials $p_n(\nu, z) = \gamma_n(\nu)z^n + \cdots$, $n = 0, 1, \ldots$, with respect to ν:

$$\int p_n(\nu, z)\overline{p_m(\nu, z)}\, d\nu(z) = \delta_{nm}.$$

We say that these polynomials have ratio asymptotic behavior if

$$\lim_{n\to\infty} p_{n+1}(\nu, z)/p_n(\nu, z) = z \tag{1}$$

uniformly on compact subsets of the exterior of the unit circle, and in this case we say shortly that ν is in the class M. If ν is given on the interval $[-1, 1]$, then the corresponding class $M(0, 1)$ is defined by the relation

$$\lim_{n\to\infty} p_{n+1}(\nu, z)/p_n(\nu, z) = z + \sqrt{z^2 - 1} \tag{2}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1, 1]$. We mention, that the right-hand sides of (1) and (2) are the corresponding conformal mappings of the outer domains of the supports onto the exterior of the unit disk, and if a ratio asymptotic exists in the outer domains, then it must be of the form (1) or (2). There is a natural one-to-one correspondence between the measures in $M(0, 1)$ and the even measures of M that is established by the usual projection of the unit circle onto $[-1, 1]$ [7].

The importance of the classes M and $M(0, 1)$ is explained by the connection between orthogonal polynomials, Padé approximation and continued fractions for Markov functions, and many investigations concerning the convergence properties of these quantities can be carried out (and have been carried out) for the case when the measure is in the class M or $M(0, 1)$. In fact, several recent investigations by A. A. Gonchar and E. A. Rahmanov indicate

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that besides Szegő's condition, the condition $\nu \in M$ or $\nu \in M(0, 1)$ seems to be the most natural condition for applications of orthogonal polynomials, see also [6].

The most general criterion for ensuring $\nu \in M$ is due to E. A. Rahmanov [7] who proved that $\nu' > 0$ almost everywhere on T implies $\nu \in M$ $(\nu \in M(0, 1))$. Here ν' denotes the Radon-Nikodym derivative of the absolutely continuous part of ν with respect to linear Lebesgue measure on $T([-1, 1])$. Several recent investigations have aimed at showing that M is a considerably larger class than is suggested by Rahmanov's result. For example, several constructions were given in [1, 3–5] to show that M contains discrete measures, and in [4] D. S. Lubinsky gave a general procedure that allowed him to show singular measures in M without point masses. In connection with these results A. A. Gonchar recently asked if Rahmanov's condition is necessary for $\nu \in M$ if we assume that the measure ν has continuous density function. Below we show that this is not the case.

The aim of this note is to prove a result which, without hardly any computation, provides us with a flexible construction of measures in the class M with certain prescribed properties. As special cases we can get the above-mentioned results concerning singular measures in M.

Theorem. *Let T be the unit circle, m the normalized arc measure on T, $r > 1$ a fixed number, and suppose that for each natural number j there is given a sequence $\{\nu_k^j\}_{k=1}^\infty$ of Borel measures on T which converges to $j^{-r}m$ in the weak*-topology. Then there exists a sequence $\{k_j\}$ such that the measure*

$$\nu = \sum_{j=1}^{\infty} \nu_{k_j}^j \tag{3}$$

is in the class M.

As an immediate corollary we get the following result which is simpler in formulation: Let $r > 1$. If $\{\nu_k\}_{k=1}^\infty$ is any sequence of measures on T converging to the arc measure in the weak*-topology, then for some sequence $\{k_j\}$ the measure

$$\nu = \sum_{j=1}^{\infty} j^{-r} \nu_{k_j}$$

belongs to the class M.

Of course, both here and in the theorem the arc measure m does not play a distinguished role. We have chosen it for convenience but it could be replaced by any other measure satisfying Szegő's condition.

Before giving the proof we mention a few possible choices for the sequences $\{\nu_k^j\}$.

(1) If each ν_k^j is a discrete measure then we get a discrete ν in M.

(2) Similarly, if each ν_k^j is a singular measure without mass point then we obtain a measure in M with the same properties.

(3) If we choose the ν_k^j in such a way that it has continuous density w_k^j with the properties $|w_k^j| \le j^{-2}$, $m(\mathrm{supp}(w_k^j)) < \varepsilon(j+1)^{-2}$, where ε is some given positive number, then we get a ν in M that has continuous density w and whose support has linear Lebesgue measure smaller than ε.

(4) As a common generalization of (1) and (2) we get the following: if μ is any measure with support T then there is a $\nu \in M$ that is absolutely continuous with respect to μ (we also mention that the support of μ must be equal to T if we want this conclusion). In fact, by choosing μ to be discrete or continuously singular, we get (1) and (2), respectively.

(5) We can mix (1)–(4) to obtain measures in M with different properties on certain arcs of T.

(6) If the measures in (1)–(5) are chosen to be even, then by taking the projection of the so constructed ν onto $[-1, 1]$ we get measures in the class $M(0, 1)$ with the above properties.

Proof. We start the proof by recalling the well-known result (see [2, 6, 7]) that $\nu \in M$ if and only if

$$\lim_{n\to\infty} \gamma_{n+1}(\nu)/\gamma_n(\nu) = 1.$$

We also need that $\gamma_n(\nu)$ is an increasing function of n and decreasing function of ν, which are immediate consequences of the formula [2]

$$\frac{1}{\gamma_n(\nu)^2} = \inf_{P_n(z)=z^n+\cdots} \int_T |P_n|^2\, d\nu ,$$

which in turn is an easy consequence of the orthogonality of the polynomials $p_k(\nu, x)$ (develop P_n into $\{p_k(\nu, \cdot)\}$).

Without loss of generality we can assume that each ν_k^j satisfies $\|\nu_k^j\| < 2j^{-r}$, where $\|\cdot\|$ indicates total mass (throw away those which do not have this property).

We select the sequence $\{k_j\}$ inductively. Suppose $k_1, \dots, k_{j-1}$ have already been selected and set

$$\mu_0^j \equiv \mu_0 = \sum_{l=1}^{j-1} \nu_{k_l}^l + \left(\sum_{l=j+1}^{\infty} \frac{1}{l^r}\right) m ,$$

$$\mu_k^j \equiv \mu_k = \mu_0^j + \nu_k^j , \qquad \mu_\infty^j \equiv \mu_\infty = \lim_{k\to\infty} \mu_k^j = \mu_0^j + \frac{1}{j^r} m.$$

We claim that there exists an N_j such that for $n \ge N_j$ we have

$$1 \le \frac{\gamma_{n+1}(\mu_k)}{\gamma_n(\mu_k)} \le 1 + \frac{30 r 2^r}{j}$$

for all $k = 0, 1, \dots, \infty$. In fact, for each k the sequence $\{\gamma_n(\mu_k)\}_{n=1}^\infty$ monotonically tends to $\gamma^*(\mu_k)$, where

$$\gamma^*(\mu_k) = \exp\left(\frac{1}{2}\int_T \log w_k\, dm\right)$$

with w_k equal to the density of the absolutely continuous component of μ_k (see [2]). We have $w_k = w_0 + \sigma_k$, where σ_k is a nonnegative measurable function on T with

$$\int_T \sigma_k\, dm \le \|\nu_k^j\| \le 2/j^r.$$

Hence we deduce from the inequality between the arithmetic and geometric means that

$$\begin{aligned}\text{(4)}\qquad \frac{\gamma^*(\mu_0)}{\gamma^*(\mu_k)} &= \exp\left(\frac{1}{2}\int \log\frac{w_k}{w_0}\,dm\right) \le \int \frac{w_k}{w_0}\,dm \\ &\le 1 + 10r2^r j^{r-1}\int \sigma_k\,dm < 1 + \frac{20r2^r}{j},\end{aligned}$$

where we used that, by the definition of μ_0,

$$w_0 \ge \frac{1}{2\pi}\left(\sum_{l=j+1}^{\infty}\frac{1}{l^r}\right) \ge \frac{1}{10r2^r j^{r-1}}.$$

Now consider the set $S = \{\mu_0, \mu_1, \dots, \mu_\infty\}$ equipped with the weak *-topology. Then S is compact. If the function f_n on S is defined by

$$f_n(\mu_k) = \min\left\{\gamma_n(\mu_k), \frac{\gamma^*(\mu_0)}{(1+20r2^r/j)}\right\},$$

then f_n is continuous on S and the sequence $\{f_n\}$ monotone increasingly converges to the constant $\gamma^*(\mu_0)/(1+20r2^r/j)$ (see (4)). Hence, the convergence is uniform on S. Seeing that $\gamma_n(\mu_k) \le \gamma_n(\mu_0) \le \gamma^*(\mu_0)$, we can finally conclude that

$$\limsup_{n\to\infty}\frac{\gamma_{n+1}(\mu_k)}{\gamma_n(\mu_k)} \le 1 + \frac{20r2^r}{j}$$

uniformly in $k \in \overline{0, \infty}$, by which the existence of N_j has been verified.

If $k_j, k_{j+1}, \dots$ all tend to infinity, then for each fixed n we have for the measure (3)

$$\gamma_n(\nu) \to \gamma_n(\mu^{j-1}_{k_{j-1}}),$$

hence we can choose numbers $K^j_j, K^j_{j+1}, \dots$ so that if $k_j \ge K^j_j$, $k_{j+1} \ge K^j_{j+1}, \dots$ then

$$\text{(5)}\qquad 1 \le \frac{\gamma_{n+1}(\nu)}{\gamma_n(\nu)} \le 1 + \frac{40r2^r}{j-1} \quad \text{for each } N_{j-1} < n \le N_j.$$

Now let k_j be equal to the maximum of the numbers $K^1_j, K^2_j, \dots, K^j_j$ (which we assume to have been defined in earlier steps of the construction with the properties above). Note that this choice of k_j does not affect the number N_j, hence we can continue our process with selecting N_{j+1}, then $K^{j+1}_{j+1}, K^{j+1}_{j+2}, \dots$, then $k_{j+1} = \max\{K^1_{j+1}, \dots, K^{j+1}_{j+1}\}$, and so on.

(5) shows that ν from (3) is in the class M.

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