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AN INEQUALITY OF ARAKI-LIEB–THIRRING (VON NEUMANN ALGEBRA CASE)

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ABSTRACT. For a trace τ on a semifinite von Neumann algebra we will prove $\tau((b^{1/2}ab^{1/2})^{rp}) \leq \tau((b^{r/2}a^rb^{r/2})^p)$. Here, $r \geq 1$, $p > 0$, and a, b are positive operators.

0. INTRODUCTION

In [1] Araki proved the following inequality:

$$\mathrm{Tr}((b^{1/2}ab^{1/2})^{rp}) \leq \mathrm{Tr}((b^{r/2}a^rb^{r/2})^p), \qquad r \geq 1, \ p > 0.$$

Here, a, b are positive operators, and Tr denotes the usual trace for operators on a Hilbert space. This inequality is a generalization of the one due to Lieb and Thirring, and closely related to the Golden–Thompson inequality (see [7, §8]). In [1] Araki asked whether the same estimate remains valid for a general trace on a semifinite von Neumann algebra. The method in [1] is to obtain a certain majorization of eigenvalues from a related operator-norm inequality (see [4]) based on the classical trick involving antisymmetric tensors. Although this trick is not available for von Neumann algebras, we show that the above inequality remains valid in the setup of general von Neumann algebras (with or without traces). We will use generalized s-numbers explained in [2, 3, 5].

1. MAJORIZATION

Throughout this article, let N be a semifinite von Neumann algebra with a faithful normal semifinite trace τ. For a τ-measurable operator x (affiliated with N), we denote the generalized s-number by $\mu_s(x)$, $s > 0$. (See [3] for details.) As in [2] we set

$$K = \left\{x \in N\colon \lim_{s\to\infty} \mu_s(x) = 0\right\},$$

which is a two-sided ideal in N. For $x \in K$ and $t \in (0, \tau(1))$ ($\tau(1)$ could be $+\infty$), we set

$$\Lambda_t(x) = \exp\left(\int_0^t \log\mu_s(x)\,ds\right).$$

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Basic properties of $\Lambda_t(\cdot)$ are

$$\begin{cases} \Lambda_t(x) = \Lambda_t(x^*) = \Lambda_t(|x|), \\ \Lambda_t(x^\alpha) = \Lambda_t(x)^\alpha \quad \text{if } \alpha > 0 \text{ and } x \text{ is positive}, \\ \Lambda_t(xy) \le \Lambda_t(x)\Lambda_t(y). \end{cases}$$

The last inequality is Theorem 2.3, [2] (see [6] for a stronger estimate) and related to the Fuglede–Kadison determinant theory. In the proof of Theorem 5.2, [2], the following is shown:

$$\Lambda_t(|ab|^{2^n}) \le \Lambda_t(a^{2^n} b^{2^n}), \tag{1}$$

$t > 0$, $n = 0, 1, 2, \dots$, and $a, b \in K_+$. When $n = 1$, this means

$$\Lambda_t(b^{1/2}ab^{1/2}) = \Lambda_t(|a^{1/2}b^{1/2}|^2) \le \Lambda_t(ab). \tag{2}$$

Although (2) is sufficient for our later purpose, the next stronger result (see [7, Theorem 8.1]) may be of independent interest.

Remark 1. Assume $x, y \in K$. If the product xy is selfadjoint, then we have $\Lambda_t(xy) \le \Lambda_t(yx)$, $t > 0$.

The following proof is a variant of that of [2, Theorem 5.2]. We may and do assume $\Lambda_t(x), \Lambda_t(y) \ne 0$ (thanks to $\Lambda_t(xy) \le \Lambda_t(x)\Lambda_t(y)$). For each $n \in \mathbb{N}_+$, we estimate

$$\begin{aligned} \Lambda_t(xy)^{2n} &= \Lambda_t(|xy|^{2n}) \\ &= \Lambda_t((xy)^{2n}) \quad (\text{since } xy \text{ is selfadjoint}) \\ &= \Lambda_t(x(yx)^{2n-1}y) \le \Lambda_t(x)\Lambda_t(yx)^{2n-1}\Lambda_t(y). \end{aligned}$$

Taking the $2n$th roots of the both sides and then letting n go to $+\infty$, we obtain the desired result.

Let us return to (1). Replacing a, b by $a^{1/2^n}$, $b^{1/2^n}$ respectively, we get

$$\Lambda_t(a^{1/2^n} b^{1/2^n}) \le \Lambda_t(ab)^{1/2^n}.$$

We would like to generalize this to

$$\Lambda_t(a^r b^r) \le \Lambda_t(ab)^r \tag{3}$$

for $r \in (0, 1]$ (and $a, b \in K_+$). For this purpose we prove the following claim: If the inequality is valid for $r = \alpha$ and $r = \beta$, $0 < \beta < \alpha \le 1$, then so is the case for $r = (\alpha + \beta)/2$. In fact, we estimate

$$\begin{aligned} \Lambda_t(a^{\frac{\alpha+\beta}{2}} b^{\frac{\alpha+\beta}{2}})^2 &= \Lambda_t(b^{\frac{\alpha+\beta}{2}} a^{\alpha+\beta} b^{\frac{\alpha+\beta}{2}}) = \Lambda_t(b^{\frac{\alpha-\beta}{2}} (b^\beta a^{\alpha+\beta} b^\beta) b^{\frac{\alpha-\beta}{2}}) \\ &\le \Lambda_t((b^\beta a^{\alpha+\beta} b^\beta) b^{\alpha-\beta}) \quad (\text{by (2) or Remark 1}) \\ &= \Lambda_t((b^\beta a^\beta)(a^\alpha b^\alpha)) \le \Lambda_t(b^\beta a^\beta)\Lambda_t(a^\alpha b^\alpha) \\ &= \Lambda_t(a^\alpha b^\alpha)\Lambda_t(a^\beta b^\beta) \le \Lambda_t(ab)^\alpha \Lambda_t(ab)^\beta \quad (\text{by the assumption}). \end{aligned}$$

By taking the square roots, we are done.

Hence, (3) is valid for r in the dense subset in $(0, 1]$. Replacing a^r, b^r by a, b respectively, we observe:

$$\Lambda_t(|ab|^r) \le \Lambda_t(a^r b^r); \qquad t > 0, \ a, b \in K_+ \tag{4}$$

for r in the dense subset in $[1, +\infty)$. We postpone generalizing these inequalities for an arbitrary r (see the paragraph after Theorem 2).

2. Approximation

Let f be a continuous increasing function on $\mathbb{R}_+$ such that $f(0) = 0$ and $t \to f(e^t)$ is convex. By the Weyl majorization theorem (see [2, Corollary 4.2]), from (4) we get

$$\int_0^t f(\mu_s(|ab|^r))\,ds \le \int_0^t f(\mu_s(a^r b^r))\,ds, \qquad t > 0. \tag{5}$$

To show (5) for an arbitrary $r \ge 1$, we choose a sequence $\{r_n\}$ for which (4) (and hence (5)) is valid. Since (for each s) the map: $x \to \mu_s(x)$ is norm-continuous [2, Proposition 1.6 (iii)], the standard argument on norm convergence and the dominated convergence theorem shows

$$\begin{cases} \int_0^t f(\mu_s(|ab|^r))\,ds = \lim_{n\to\infty} \int_0^t f(\mu_s(|ab|^{r_n}))\,ds, \\ \int_0^t f(\mu_s(a^r b^r))\,ds = \lim_{n\to\infty} \int_0^t f(\mu_s(a^{r_n} b^{r_n}))\,ds. \end{cases}$$

Hence (5) is valid for an arbitrary $r \ge 1$ (and $a, b \in K_+$).

Next we would like to generalize (5) for positive τ-measurable operators a, b satisfying $\lim_{s\to\infty} \mu_s(a) = \lim_{s\to\infty} \mu_s(b) = 0$. Let $a = \int_0^\infty \lambda\, de_\lambda$ and $b = \int_0^\infty \lambda\, df_\lambda$ be the spectral decompositions. We set

$$a_n = \int_{n^{-1}}^n \lambda\, de_\lambda, \qquad b_n = \int_{n^{-1}}^n \lambda\, df_\lambda \qquad (n \in \mathbb{N}_+).$$

(See [3, Proposition 3.2].) Fixing $m \in \mathbb{N}_+$, we estimate

$$\begin{aligned} \int_0^t f(\mu_s(|ab_m|^r))\,ds &= \int_0^t f(\mu_s(b_m a^2 b_m)^{r/2})\,ds \\ &= \sup_n \int_0^t f(\mu_s(b_m a_n^2 b_m)^{r/2})\,ds \\ &\quad \left(\begin{array}{l} \text{Since } a_n^2 \uparrow a^2 \text{ in measure, [3, Lemma 3.4] shows} \\ \mu_s(b_m a_n^2 b_m) \uparrow \mu_s(b_m a^2 b_m). \text{ Hence the result follows} \\ \text{from the monotone convergence theorem.} \end{array}\right) \\ &= \sup_n \int_0^t f(\mu_s(|a_n b_m|^r))\,ds \\ &\le \sup_n \int_0^t f(\mu_s(a_n^r b_m^r))\,ds \quad (\text{since } a_n, b_m \in K_+) \\ &= \sup_n \int_0^t f(\mu_s(b_m^r a_n^{2r} b_m^r)^{1/2})\,ds \\ &= \int_0^t f(\mu_s(b_m^r a^{2r} b_m^r)^{1/2})\,ds \\ &\qquad\qquad (\text{as before since } a_n^{2r} \uparrow a^{2r} \text{ in measure}) \\ &= \int_0^t f(\mu_s(a^r b_m^r))\,ds. \end{aligned}$$

Notice $\mu_s(|ab_m|^r) = \mu_s(ab_m)^r = \mu_s(b_m a)^r = \mu_s(|b_m a|^r)$ and $\mu_s(a^r b_m^r) = \mu_s(b_m^r a^r)$. Hence, when m goes to $+\infty$, we can use the monotone convergence as in the preceding argument and get (5) for a, b in the wider class.

3. Main results

In the §§1, 2 we have shown

Theorem 2 [1, Theorem 2]. *Let $r \geq 1$ and f be a continuous increasing function on $\mathbb{R}_+$ such that $f(0) = 0$ and $t \to f(e^t)$ is convex. For positive τ-measurable operators a, b satisfying $\lim_{s\to\infty} \mu_s(a) = \lim_{s\to\infty} \mu_s(b) = 0$, we have*

$$\int_0^t f(\mu_s(|ab|^r))\, ds \leq \int_0^t f(\mu_s(a^r b^r))\, ds, \qquad t > 0.$$

When $f(t) = t^p$, $p > 0$, the theorem means

$$\int_0^t \mu_s(|ab|^{rp})\, ds \leq \int_0^t \mu_s(|a^r b^r|^p)\, ds, \qquad t > 0. \tag{6}$$

For a τ-measurable operator x satisfying the "Lorentz space condition" ($\mu_s(x) \leq Cs^{-\alpha}$ for some C, $\alpha > 0$), the expression $\Lambda_t(x) = \exp(\int_0^t \log \mu_s(x)\, ds)$ still makes sense (see [3, p. 286]). For such positive operators a, b, (6) obviously implies

$$\left\{\int_0^t \mu_s(|ab|^r)^p \frac{ds}{t}\right\}^{1/p} \leq \left\{\int_0^t \mu_s(a^r b^r)^p \frac{ds}{t}\right\}^{1/p}.$$

Letting $p \downarrow 0$ (see [3, p. 288]), we conclude that (4) is valid for an arbitrary $r \in [1, +\infty)$, or equivalently, (3) is valid for an arbitrary $r \in (0, 1]$.

Letting $t \uparrow +\infty$ in (6) (see [3, Corollary 2.8]), we get the following (semifinite) von Neumann algebra version of the inequality in the introduction:

Corollary 3. *For positive operators a, b in the previous theorem we have*

$$\tau(|ab|^{rp}) \leq \tau(|a^r b^r|^p); \qquad r \geq 1, \ p > 0.$$

Let $\|x\|_p = \tau(|x|^p)^{1/p}$, $p > 0$, be the L^p (quasi-)norm. Then the corollary means $\| |ab|^r \|_p \leq \|a^r b^r\|_p$. Let M be a general (not necessarily semifinite) von Neumann algebra. We have (for example Haagerup) L^p-spaces $L^p(M)$, $p > 0$, and $\|\cdot\|_p$ still makes sense. (See [3, 1.6; 5] for quick introduction to the subject.)

Theorem 4. *Let $r \geq 1$ and a, b be positive elements in $L^{p_1 r}(M)$ and $L^{p_2 r}(M)$ ($p_i > 0$) respectively. For $p > 0$ satisfying $p^{-1} = p_1^{-1} + p_2^{-1}$ (hence $ab \in L^{pr}(M)$ and $|ab|^r$, $a^r b^r \in L^p(M)$) we have*

$$\| |ab|^r \|_p \leq \|a^r b^r\|_p.$$

Proof. We will use the trick used in the appendix to [5]. Let N be the crossed product $M \rtimes_\sigma \mathbb{R}$ relative to a modular automorphism group, and τ be the canonical trace on N scaled (in the usual way) by the dual action. From the definition, a, b are τ-measurable operators (affiliated with N), and [3, Lemma 4.8] says

$$\begin{cases} \mu_s(ab) = \|ab\|_{pr} s^{-1/pr}, \\ \mu_s(a^r b^r) = \|a^r b^r\|_p s^{-1/p}. \end{cases}$$

Therefore, (6) (with $t = 1$ and $p/2$ instead of p) implies that

$$\int_0^1 (\|ab\|_{pr} s^{-1/pr})^{pr/2}\, ds \le \int_0^1 (\|a^r b^r\|_p s^{-1/p})^{p/2}\, ds .$$

Evaluating the integrals, we obtain $2\|ab\|_{pr}^{pr/2} \le 2\|a^r b^r\|_p^{p/2}$, that is, $\|ab\|_{pr}^r \le \|a^r b^r\|_p$. However, the left side is obviously equal to $\| \, |ab|^r \|_p$. Q.E.D.

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