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SOME BANACH ALGEBRAS WITHOUT DISCONTINUOUS DERIVATIONS

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ABSTRACT. It is shown that the completion of $A(G)$ in either the multiplier norm or the completely bounded multiplier norm is a Banach algebra without discontinuous derivations when G is either F_2 or $\mathrm{SL}(2, \mathbb{R})$.

1. INTRODUCTION

In [6], it was shown that a locally compact group G is amenable if and only if every derivation D from $A(G)$, the Fourier algebra of G, into an arbitrary Banach $A(G)$-bimodule X is continuous. The same result can be shown to hold for the Herz algebras $\mathrm{Ap}(G)$, $1 < p < \infty$ [7]. Since the class of amenable groups is substantial (it includes all compact groups and all commutative groups) a large number of Banach algebras without discontinuous derivations have been identified. However, since $\mathrm{SL}(2, \mathbb{R})$ and F_2, the free group on two generators, are nonamenable, $A(\mathrm{SL}(2, \mathbb{R}))$ and $A(F_2)$ have discontinuous derivations. We will show that both $A(\mathrm{SL}(2, \mathbb{R}))$ and $A(F_2)$ can be given natural norms in such a way that their completions will be Banach algebras without discontinuous derivations.

2. PRELIMINARIES AND NOTATION

Let G be a locally compact group. Let $A(G)$ be the Fourier algebra of G as defined by P. Eymard in [4]. $A(G)$ is a Banach algebra with respect to pointwise multiplication. It is also the predual of the von Neumann algebra $\mathrm{VN}(G)$ associated with the left-regular representation of G on $L^2(G)$ and a closed ideal in $B(G)$, the Fourier Stieltjes algebra of G.

Let $MA(G)$ denote the space of multipliers of $A(G)$. That is, the continuous functions ψ on G such that $\psi u \in A(G)$ for every $u \in A(G)$. For each $\psi \in MA(G)$, $u \in A(G)$, let $m_\psi(u) = \psi u$. Denote by $\|\psi\|_m$ the operator norm of m_ψ. We call ψ a completely bounded multiplier of $A(G)$ if m_ψ^*, the adjoint of m_ψ, is a completely bounded map on $\mathrm{VN}(G)$ [3]. Let $\|\psi\|_{M_0}$ be the

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completely bounded norm of m_ψ^* and let $M_0A(G)$ denote the Banach algebra of completely bounded multipliers of $A(G)$. Then $B(G) \subseteq M_0A(G) \subseteq MA(G)$ and $\|u\|_M \leq \|u\|_{M_0} \leq \|u\|_{B(G)}$ for every $u \in B(G)$.

We will denote by $A_M(G)$ and $A_{M_0}(G)$ the closure of $A(G)$ in $MA(G)$ and in $M_0A(G)$ respectively. It is well known that G is amenable if and only if $MA(G) = B(G)$ [12, 13]. In this case $\|u\|_{B(G)} = \|u\|_M$ for every $u \in B(G)$ and hence $A(G) = A_M(G) = A_{M_0(G)}$.

If $\mathscr{A}$ is a Banach algebra, then a derivation D on $\mathscr{A}$ is a linear map from $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule X such that $D(uv) = u \cdot D(v) + D(u) \cdot v$ for every u, $v \in \mathscr{A}$. If $\mathscr{A}$ is commutative, we will denote the maximal ideal space of $\mathscr{A}$ by $\Delta(\mathscr{A})$. We also use $\Delta(\mathscr{A})$ to denote the multiplicative linear functionals which correspond to the maximal ideal. Given a closed subset A of $\Delta(\mathscr{A})$, we denote by $I_{\mathscr{A}}(A)$, the ideal $\{u \in \mathscr{A}\,;\, u(x) = 0 \text{ for every } x \in A\}$, where $\mathscr{A}$ is realized as an algebra of functions on $\Delta(\mathscr{A})$ by means of the Gelfand transform. A is called a set of spectral synthesis for $\mathscr{A}$ or simply an S-set if $I^0_{\mathscr{A}}(A) = \{u \in \mathscr{A}\,;\, \operatorname{supp} u \text{ is compact, } \operatorname{supp} u \cap A = \varnothing\}$ is dense in $I_{\mathscr{A}}(A)$.

3. Spectral synthesis and automatic continuity of derivations

Lemma 1. *Let G be a locally compact group. Then*

$$\Delta(A_M(G)) = \Delta(A_{M_0}(G)) = G.$$

Proof. Let $\psi \in \Delta(A_M(G))$. Then $\psi|_{A(G)}$, the restriction of ψ to $A(G)$, belongs to $\Delta(A(G))$. By [4, p. 222] there exists an $x \in G$ such that $\psi(u) = u(x)$ for every $u \in A(G)$.

Let $v \in A_M(G)$. As $A(G)$ is dense in $A_M(G)$, we can find $\{u_k\} \subseteq A(G)$ such that $\|u_k - v\|_M \to 0$. However, convergence in $A_M(G)$ implies convergence in the topology of uniform convergence on compacta and hence in the topology of pointwise convergence. Therefore $\psi(v) = \lim_k \psi(u_k) = \lim_k u_k(x) = v(x)$. It follows that as a set $\Delta(A_M(G)) = G$.

Now suppose that $\{x_\alpha\}_{\alpha \in I}$ is a net in G such that x_α converges to $x \in G$ in the usual topology on G. Then for each $u \in A_M(G)$, $u(x_\alpha) \to u(x)$. Hence $x_\alpha \to x$ in the $\sigma(A_M(G)^*, A_M(G))$ topology. Conversely, if $x_\alpha \to x$ in the $\sigma(A_M(G)^*, A_M(G))$ topology, then for every $u \in A(G)$, $u(x_\alpha) \to u(x)$. Since $G \subseteq A(G)^*$, $x_\alpha \to x$ in the $\sigma(A_M(G)^*, A_M(G))$ topology. But $\Delta(A(G))$ is homeomorphic to G so $x_\alpha \to x$ in G. Therefore the $\sigma(A_M(G)^*, A_M(G))$ topology agrees with the usual topology on G.

A similar argument establishes that $\Delta(A_{M_0}(G)) = G$. □

Proposition 1. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. Then $\{e\}$ is an S-set of $\mathscr{A}$.*

Proof. Let $v \in \mathscr{A}$ with $v \in I_{\mathscr{A}}(\{e\})$. Then there exists $u_k \in A(G)$ with $\|u_k - v\|_{\mathscr{A}} \to 0$. Furthermore since $u_k(e) \to v(e) = 0$, we can assume that $|u_k(e)| < 1/2k$. We can also find $w_k \in A(G)$ with $\|w_k\|_{A(G)} = w_k(e) = u_k(e)$. Let $v_k = u_k - w_k$. Then $v_k(e) = 0$. Since $v_k \in I_{A(G)}(\{e\})$ and $\{e\}$ is an S-set for $A(G)$ [4, p. 229], there exists $z_k \in A(G)$ with $\|z_k - v_k\|_{A(G)} \leq 1/2k$,

$\operatorname{supp} z_k$ is compact and $\operatorname{supp} z_k \cap \{e\} = \varnothing$. However

$$\begin{aligned}\|z_k - v\|_{\mathscr{A}} &\le \|z_k - v_k\|_{\mathscr{A}} + \|v_k - v\|_{\mathscr{A}} \\ &\le \|z_k - v_k\|_{A(G)} + \|u_k - v\|_{\mathscr{A}} + \|w_k\|_{A(G)} \\ &\le \|u_k - v\|_{\mathscr{A}} + 1/k.\end{aligned}$$

Hence $\|z_k - v\|_{\mathscr{A}} \to 0$. Therefore $I^0_{\mathscr{A}}(\{e\})$ is dense in $I_A(\{e\})$ and $\{e\}$ is an S-set. □

Proposition 2. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathscr{A}$ has a bounded approximate identity, then so does $I_{\mathscr{A}}(\{e\})$.*

Proof. Let $\{u_\alpha\}_{\alpha\in I}$ be a bounded approximate identity in $\mathscr{A}$. Let $\mathscr{F}(\{e\}) = \{K \subset G;\ K \text{ is compact, } K \cap \{e\} = \varnothing\}$. For every $K \in \mathscr{F}(\{e\})$, there exists $v_K \in B(G)$ such that $\|v_K\|_{\mathscr{A}} \le \|v_K\|_{B(G)} = 1$, $u_K(e) = 1$, and $v_K(x) = 0$ for every $x \in K$. Define $w_{K,\alpha} \in \mathscr{A}$ by $w_{K,\alpha} = u_\alpha - v_K u_\alpha$. Then $w_{K,\alpha}(e) = 0$, $\|w_{K,\alpha}\|_{\mathscr{A}} \le 2\|u_\alpha\|_{\mathscr{A}}$, and $w_{K,\alpha} v = u_\alpha v$ for every $v \in \mathscr{A}$ with $\operatorname{supp} v \subset K$.

Order $K \times I$ by $(K_1, \alpha_1) \le (K_2, \alpha_2)$ if and only if $K_1 \subseteq K_2$ and $\alpha_1 \le \alpha_2$. If $v \in \mathscr{A}$ and $\operatorname{supp} v \in \mathscr{F}(\{e\})$, then $v = \lim_{K,\alpha} w_{K,\alpha} v$. By Proposition 1, such v's are dense in $I_{\mathscr{A}}(\{e\})$. As $\{w_{K,\alpha}\}_{K\times I}$ is bounded, $\lim_{K,\alpha} w_{K,\alpha} w = w$ for every $w \in I_{\mathscr{A}}(\{e\})$. □

The proof of Proposition 2 is a modification of the proof of [7, Proposition 3.2]. The case where G is amenable is due to A. Lau [11, Corollary 4.11].

Corollary 1. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathscr{A}$ has a bounded approximate identity, then $\mathscr{A}$ satisfies Ditkin's condition. Furthermore, if A has a closed subset of G and the boundary of A contains no nontrivial perfect set, then A is an S-set. In particular, every finite subset of G is an S-set for $\mathscr{A}$.*

Proof. Let $x \in G$ and $u \in \mathscr{A}$ be such that $u(x) = 0$. From the proof of Proposition 2 (translate is $x \ne e$), we see there exists a sequence $\{v_n\} \subset \mathscr{A}$ for which each v_n vanishes in a neighborhood V_n of $\{x\}$ and $\lim_n \|v_n u - u\|_{\mathscr{A}} = 0$.

Assume that G is not compact. Let $u \in \mathscr{A}$, $u \ne 0$. Let $\varepsilon > 0$. If $\{u_\alpha\}_{\alpha\in I}$ is a bounded approximate identity in $\mathscr{A}$, then there exists u_{α_ε} such that $\|u_{\alpha_\varepsilon} u - u\|_{\mathscr{A}} < \varepsilon/2$. We can find $v \in A(G)$ with $\operatorname{supp} v$ compact such that $\|u_{\alpha_\varepsilon} - v\|_{\mathscr{A}} \le \varepsilon/2\|u\|_{\mathscr{A}}$. Then $\|vu - u\|_{\mathscr{A}} \le \|vu - u_\alpha u\|_{\mathscr{A}} + \|u_\alpha u - u\|_{\mathscr{A}} \le \varepsilon$.

Hence $\mathscr{A}$ satisfies Ditkin's condition [9, p. 49]. The remaining statements follow immediately from Ditkin's theorem [9, p. 497]. □

Corollary 1 is simply [7, Lemma 5.2, Proposition 5.3] when G is assumed to be amenable.

Proposition 3. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathscr{A}$ has a bounded approximate identity and I is a closed cofinite ideal in $\mathscr{A}$ (i.e., $\dim \mathscr{A}/I < \infty$), then $I = I(A)$ for some finite set $A = \{x_1, \dots, x_n\}$ where n is the codimension of I.*

Proof. Let $A = Z(I) = \{x \in G;\ u(x) = 0 \text{ for every } u \in I\}$. Let $n = \operatorname{codim} I$. Assume that A contains $n+1$ distinct elements $\{x_1, \dots, x_{n+1}\}$. We can find a compact neighborhood V_k of each x_k such that $V_j \cap V_k = \varnothing$ if $j \ne k$. We can also find $u_k \in A(G)$ such that $\operatorname{supp} u_k \subseteq V_k$ and $u_k(x_k) = 1$ for $1 \le k \le n+1$. But if $\psi: \mathscr{A} \to \mathscr{A}/I$ is the canonical homomorphism, then $\{\psi(u_1), \dots, \psi(u_{n+1})\}$ is a linearly independent subset, which is impossible if codim $I = n$. Therefore A has at most n elements. Since A is finite, it is an S-set by Corollary 1. Therefore $I = I(A)$.

Let $A = \{x_1, \dots, x_k\}$. Let $u_k \in A(G)$ be such that $u_k(x_k) = 1$ and $u_k(x_j) = 0$ if $j \neq k$. Let $u \in \mathscr{A}$. Then

$$u = \sum_{i=1}^{k} u(x_i)u_i + \left(u - \sum_{i=1}^{k} u(x_i)u_i\right).$$

Since

$$u - \sum_{i=1}^{k} u(x_i)u_i \in I(A), \qquad k \geq n.$$

Hence $k = n$. □

Proposition 4. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. Assume that $\mathscr{A}$ has a bounded approximate identity. Let I be a closed cofinite ideal of $\mathscr{A}$. Then I has a bounded approximate identity. In particular $I^2 = \{\sum_{i=1}^n u_i v_i,\ u_i, v_i \in I\} = I$.*

Proof. By Proposition 3, $I = I(\{x_1, \dots, x_n\})$ for some finite subset of G.

Let A, B be two closed subsets of G. Assume that $I_{\mathscr{A}}(A)$ and $I_{\mathscr{A}}(B)$ have bounded approximate identities $\{u_i\}_{i\in T}$, $\{v_j\}_{j\in J}$ respectively. Then it is easy to see that $\{u_i v_j\}_{T\times J}$ is a bounded approximate identity for $I(A \cup B)$.

Since $\mathscr{A}$ has a bounded approximate identity, Proposition 2 implies that $I_{\mathscr{A}}(\{e\})$ has a bounded approximate identity. By translating, we see that $I_{\mathscr{A}}(\{x\})$ has a bounded approximate identity for every $x \in G$. A simple induction argument shows that I also has a bounded approximate identity. Then $I^2 = I$ follows from Cohen's factorization theorem [9, p. 268]. □

Theorem 1. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathscr{A}$ has a bounded approximate identity, then I is a cofinite ideal of $\mathscr{A}$ if and only if $I = I(A)$ for some finite subset A of G.*

Proof. Proposition 4 shows that every closed cofinite ideal of $\mathscr{A}$ is idempotent. By [2, Theorem 2.3] every cofinite ideal of $\mathscr{A}$ must be closed and is therefore of the form $I(A)$ for some finite subset A of G by Proposition 3.

If $I = I(A)$ for a finite subset A of G, then the proof of Proposition 3 shows that I is cofinite. □

If G is amenable, then Proposition 3 and Theorem 1 follow from [7, Corollary 5.6, Theorem 5.8].

Theorem 2. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. If $\mathscr{A}$ has a bounded approximate identity, then every homomorphism from $\mathscr{A}$ with finite-dimensional range is continuous.*

Proof. The statement follows immediately from Theorem 1, Proposition 4, and [2, Theorem 2.3]. □

Lemma 2. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. Assume that $\mathscr{A}$ has a bounded approximate identity. Let I be a closed ideal in $\mathscr{A}$ with infinite codimension. Then there exist sequences $\{u_n\}$, $\{v_n\}$ in $A(G)$ such that $u_n v_1 \cdots v_{n-1} \notin I$ but $u_n v_1 \cdots v_n \in I$ for $n \geq 2$.*

Proof. If I has infinite codimension, then $A = Z(I) = \{x \in G,\ u(x) = 0\ \forall u \in I\}$ must be infinite by Theorem 1.

From the proof of [6, Lemma 2], we see that we can find sequences $\{u_n\}$, $\{v_n\} \subseteq A(G)$ such that $u_n v_1 \cdots v_{n-1} \notin I_{A(G)}(A)$ while $u_n v_1 \cdots v_n = 0$. Since $I_{A(G)}(A) \subseteq I_{\mathscr{A}}(A)$, the result follows. $\square$

Theorem 3. *Let $\mathscr{A}$ be either $A_M(G)$ or $A_{M_0}(G)$. Assume that $\mathscr{A}$ has a bounded approximate identity. Then every derivation D of $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule X is continuous.*

Proof. This follows immediately from Proposition 4, Lemma 2, and [10, Theorem 2]. $\square$

For the algebra $A(G)$, the automatic continuity properties listed in Theorems 2, 3 are characteristic of amenable groups. We see that if $A_M(G)$ or $A_{M_0}(G)$ has a bounded approximate identity, then it possesses many of the properties of the Fourier algebra of an amenable group.

In [5] it was shown that $A(F_2)$ has an approximate identity which is necessarily unbounded in $A(G)$ but is bounded in $\|\cdot\|_M$. Hence for the prototypical nonamenable group F_2, $A_M(F_2)$ has a bounded approximate identity.

In [3] it was shown that the bounded approximate identity in $A_M(F_2)$ is also bounded in $\|\cdot\|_M$. Moreover, if $G = \mathrm{SL}(2, \mathbb{R})$, $G = \mathrm{SO}(n, 1)$, or G is any closed subgroup of any of these groups, then $A(G)$ has an approximate identity bounded in $\|\cdot\|_M$. Therefore we have:

Theorem 4. *Let G be $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{SO}(n, 1)$, or F_n for $n = 2, 3, \ldots$. Let $\mathscr{A}$ be either $A_M(g)$ or $A_{M_0}(G)$. Then every homomorphism for A with finite-dimensional range is continuous and every derivation from $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule is continuous.*

Proof. This follows immediately from Theorem 2, Theorem 3, and [3, Theorem 3.7]. $\square$

De Canniere and Haagerup had speculated about the existence of a bounded approximate identity in $A_M(G)$ and $A_{M_0}(G)$ for any locally compact group [3]. This would have provided a large new class of Banach spaces without discontinuous derivations, however, it has recently been shown that this is not the case (see [1]).

It would be of interest to know whether the above automatic continuity properties are characteristic of those groups for which either $A_M(G)$ or $A_{M_0}(G)$ has a bounded approximate identity.

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