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BLOCKS OF SMALL DEFECT

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ABSTRACT. A group of odd order G with $O_p(G) = 1$ has a block of defect less than $[n/2]$, where $p^n = |G|_p$. In addition, if G is supersolvable by nilpotent, G has a block of defect zero.

1. INTRODUCTION

It is an interesting problem to give necessary and sufficient conditions for the existence of p-blocks of defect zero. If a finite group G has a block of defect zero, it is well known that $O_p(G) = 1$, although, of course, this is not a sufficient condition. Ito proved [3, X.6.5] that if G is a nilpotent by nilpotent group of odd order, then G has a block of defect zero iff $O_p(G) = 1$.

Recent results on regular orbit theorems have associated conditions for the existence of blocks of defect zero [1]. For groups of odd order, with $O_p(G) = 1$, the exceptions are essentially nilpotent by supersolvable as Theorem 1 of [1] shows us.

In general, we try to find the smallest defect $d(B)$ of a block B of G. This is given in Theorem A below.

Theorem A. *Let G be a (solvable) group of odd order such that $O_p(G) = 1$ and $|G|_p = p^n$. Then G contains a p-block B such that $d(B) \leq [n/2]$. The bound is best possible.*

It is not true in general that there exists a block B with $d(B) \leq [n/2]$, as $G = A_7$ $(p = 2)$ shows us.

By work of Michler and Willems [7, 8] every simple group except possibly the alternating group has a block of defect zero for $p \geq 5$. Perhaps the following has an affirmative answer.

Question. If G is a finite group with $O_p(G) = 1$, $p \geq 5$, and $|G|_p = p^n$, does G contain a block of defect less than $[n/2]$?

If $\mathscr{F}$ is the class of supersolvable by nilpotent groups, we can prove the following.

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Theorem B. *Let $G \in \mathscr{F}$ with $|G|$ odd. Then G has a p-defect zero character if and only if $O_p(G) = 1$.*

2. Blocks of defect less than $[n/2]$

The next lemma is a key tool for proving Theorem A. We denote by $F(G)$ the Fitting subgroup of the group G.

2.1. **Lemma.** *Let G be a group of odd order and let V be a faithful and irreducible KG module,* char(K) *being odd. Then there exists a normal subgroup H of G (possibly $H = 1$) and a vector $v \in V$ such that: $C_G(v)^\# \subseteq H - F(H)$ and $H/F(H)$ is abelian.*

Proof. See the proof of Theorem 3.1 of [2].

Theorem A. *Let G be a (solvable) group of odd order such that $O_p(G) = 1$ and $|G|_p = p^n$. Then G contains a p-block B such that $d(B) \leq [n/2]$. The bound is best possible.*

Proof. Induction on $|G|$. Consider $\bar{G} = G/\Phi(G)$. As $F(G/\Phi(G)) = F(G)/\Phi(G)$, we have that $O_p(\bar{G}) = 1$ and $|G|_p = |\bar{G}|_p$. If $\Phi(G) \neq 1$, then the result is true for $\bar{G}$. Let $\bar{B}$ be a p-block of $\bar{G}$ such that $d(\bar{B}) \leq [n/2]$. By Lemma V.4.3. of [3], there exists a p-block B of G such that $d(B) = d(\bar{B})$. Hence we may assume that $\Phi(G) = 1$.

Now, $V = \mathrm{Irr}(F(G))$ is a faithful and completely reducible $G/F(G)$-module (over different fields, possibly).

Put $V = V_1 \oplus \cdots \oplus V_t$, where each V_i is an irreducible G-module.

Define $K_i = C_G(V_i)$, $\bar{G}_i = G/K_i$ and use the bar convention. By the lemma above, there exists a normal subgroup H_i of G containing K_i and an element $\lambda_i \in V_i$ such that $C_{\bar{G}_i}(\lambda_i)^\# \subseteq \bar{H}_i - F(\bar{H}_i)$. Furthermore, $\bar{H}_i/F(\bar{H}_i)$ is abelian.

Consider $\lambda = \lambda_1 \times \cdots \times \lambda_t$ and put $C = C_G(\lambda)$.

We may view $G/F(G)$ as a subgroup of $\bar{G}_1 \times \cdots \times \bar{G}_t$. This shows that $C \subseteq F_3(G)$ and $C \cap F_2(G) = F(G)$, where, as usual, $F_1(G) = F(G)$, $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$.

We consider separately two cases.

(1) $|C|_p \leq {|G|_p}^{1/2}$.

Take $\chi \in \mathrm{Irr}(G)$ lying over λ and let B be the p-block of G containing χ. As $F(G)$ is a p'-group, Lemma V.2.3 of [3] shows that every irreducible character ψ in B has λ as an irreducible constituent. Now $\psi(1)_p \geq |G : I_G(\lambda)|_p \geq {|G|_p}^{1/2}$.

The result follows.

(2) $|C|_p > {|G|_p}^{1/2}$.

Now $|F_3(G)/F_2(G)|_p > {|G|_p}^{1/2}$.

Let $P/F_2(G)$ be a Sylow p-subgroup of $F_3(G)/F_2(G)$. Where $Y = O_{p'}(F(G/F(G))$, observe that $W = \mathrm{Irr}(Y/\Phi(Y))$ is a faithful and completely reducible $P/F_2(G)$-module.

By Gow's regular orbit theorem [4, 2.6], we have $\mu \in W$ such that $C_P(\mu) = F_2(G)$. We may view μ as a character of the preimage X of Y in G. Observe that X is a p'-group. Take $\chi \in \mathrm{Irr}(G)$ lying over μ. Now χ lies over an irreducible character ψ of P lying over μ. Clearly, $\psi(1)_p \geq |F_3(G)/F_2(G)|_p >$

$|G|_p{}^{1/2}$. As P is normal in G, we have $\chi(1)_p \geq \psi(1)_p$. Now the same argument as in case (1) completes the proof.

To show that the bound is best possible, consider two odd primes p and q such that $q \equiv 1 \pmod{p}$. Now $(q^p-1)/q-1$ is odd and $(q^p-1)/q-1)_p = p$. Now $C_p \ltimes GF(q^p)^*$ acts on $GF(q^p)^+$, C_p acting as a Galois automorphism. Thus $H = C_p \ltimes C_{(q^p-1/g-1)}$ acts on $V = C_q \times \cdots \times C_q$ (p times) and V does not contain any regular H-orbit. Consider $X = H \ltimes V$ and let G be the direct product of m copies of X. Then, if $\chi \in \operatorname{Irr}(G)$, we have $\chi(1)_p \leq p^m$. Thus $d(B) \geq m$ for any p-block of G and $|G|_p = p^{2m}$.

3. Supersolvable by nilpotent groups

We need a regular orbit theorem for proving Theorem B. It is a slight generalization of [4, 2.6]

3.1. **Theorem.** *Let G be a group of odd order and let V be a faithful irreducible KG-module, where K is a field of odd characteristic q. Assume that G is p-nilpotent for some prime $p \neq q$ and that*

$$V_{O_{p'}(G)} = V_1 \oplus \cdots \oplus V_t,$$

where each V_i is a 1-dimensional $KO_{p'}(G)$-module. Then there exists $v \in V$ such that $C_G(v) = 1$.

Proof. We induct on $|G|$.

Write $H = O_{p'}(G)$. We claim that V_H is not homogeneous. Let W be an irreducible submodule of V_H and let $I = \{g \in G \mid Wg \cong W \text{ as } KH\text{-modules}\}$. By hypothesis, $\dim_K W = 1$.

If $I = G$, then W is a faithful 1-dimensional KH-module and V is KH-homogeneous.

A well-known argument implies that H is contained in $Z(G)$. Then G is nilpotent and since V is faithful, q does not divide $|G|$. Now, Gow's regular orbit theorem [4. 2.6] gives us $v \in V$ such that $C_G(v) = 1$.

Thus, we may assume that V_H is not homogeneous. Then, we may find a normal subgroup N of G of index p and an irreducible KN-module U with $U^G = V$ such that W is an irreducible submodule of U_H. By Clifford's theorem, we may apply induction to N and find $u \in U$ such that

$$C_N(u) = C_N(U).$$

If $x_1, \ldots, x_p$ is a set of coset representatives of N in G, by using the facts that $nu \neq -u$ $\forall n \in N$ (because q and $|G|$ are odd) and that $\operatorname{core}_G(C_N(u)) = C_G(V) = 1$, it can be checked that $C_G(v) = 1$, where

$$v = u \otimes x_1 + \cdots + u \otimes x_{p-1} - u \otimes x_p.$$

This finishes the proof of the theorem.

Recall that a group G has a p-block of defect zero if it has an irreducible character χ with $\chi(1)_p = |G|_p$.

3.2. **Lemma.** (a) *Let H be a subgroup of G and let $\mu \in \operatorname{Irr}(H)$, with $\mu^G = \chi \in \operatorname{Irr}(G)$. Then $\chi(1)_p = |G|_p$ if and only if $\mu(1)_p = |H|_p$.*

(b) *Let N be a normal subgroup of G, let $\theta \in \operatorname{Irr}(N)$ and let $\chi \in \operatorname{Irr}(G \mid \theta)$.*

If $\chi(1)_p = |G|_p$ *, then* $\theta(1)_p = |N|_p$ *. Conversely, if* $\theta(1)_p = |N|_p$ *, and* $|G:N|$ *is a* p'*-number, then* $\chi(1)_p = |G|_p$ *.*

Proof. (a) Since $|G:H|\mu(1) = \chi(1)$, we have that

$$|G|_p/\chi(1)_p = |H|_p/\mu(1)_p .$$

(b) By (a) and the Clifford correspondence, we may assume that $\chi_N = u\theta$, for some integer u. Thus $|G|_p = u_p\theta(1)_p$.

We know that $\theta(1)_p$ divides $|N|_p$, and that u_p divides $|G/N|_p$ [6, 11.29]. Since $|G/N|_p|N|_p = |G|_p$, necessarily, we will have

$$u_p = |G/N|_p \quad \text{and} \quad \theta(1)_p = |N|_p .$$

Conversely, suppose that $\theta(1)_p = |N|_p$, and that $|G:N|$ is a p'-number. Let $T = I_G(\theta)$, let $\mu \in \mathrm{Irr}(T)$ the Clifford correspondent of χ over θ, and write $\mu_N = v\theta$, for some integer v. Then v is a p'-number,

$$\mu(1)_p = \theta(1)_p = |N|_p = |T|_p ,$$

and by (a) the proof is complete. Now we can prove Theorem B.

Theorem B. *Let* $G \in \mathscr{F}$ *with* $|G|$ *odd. Then* G *has a* p*-defect zero character if and only if* $O_p(G) = 1$ *.*

Proof. If $\chi \in \mathrm{Irr}(G)$ with $\chi(1)_p = |G|_p$, let θ be an irreducible constituent of $\chi_{O_p(G)}$. By part (b) of the lemma above, $\theta(1) = O_p(G)$, and thus $O_p(G) = 1$.

Suppose that $G \in \mathscr{F}$ with $|G|$ odd. We prove that G has a p-defect zero character by induction on $|G|$.

Let N be a normal supersolvable subgroup of G such that G/N is nilpotent. Let P be a Sylow p-subgroup of G and let $F = F(G)$ be the Fitting subgroup of G.

First we prove that G/N is a p-group. Since G/N is nilpotent, PN is a normal subgroup of G. Since $O_p(PN) = 1$, by induction and the lemma, we may assume that $PN = G$.

We claim that G is p-nilpotent with $O_{p'}(G)$ supersolvable. Let H be a Hall p'-subgroup of G. Since G/N is a p-group, H is contained in N. Since N is supersolvable, $N/F(N)$ is abelian. But $F(N)$ is a p'-group, because $O_p(F(N)) = 1$. Thus $F(N) \subseteq H \subseteq N$.

This implies that H is normal in G, as wanted.

Now, since $F/\Phi(G) = F(G/\Phi(G))$, $O_p(G/\Phi(G)) = 1$, and we may assume that $\Phi(G) = 1$.

Write $F = E_1 \times \cdots \times E_s$, where the E_i's are minimal normal subgroups of G.

If E is any normal subgroup of G contained in F, we claim that there exists $\lambda \in \mathrm{Irr}(E)$ such that $I_G(\lambda) = C = C_G(E)$.

Suppose that $|E|$ is a q-power, for a prime q. Observe that $E \subseteq H$. Since H is supersolvable, by Clifford's theorem E_H is a direct sum of 1-dimensional KE-submodules, and so it is $E_{HC/C}$, $K = GF(q)$.

Let $\hat{E}$ =Irr(E). Then $\hat{E}$ is a faithful irreducible $K[G/C]$-module.

Since $E_{HC/C} = X_1 \oplus \cdots \oplus X_t$, where the X_i's are 1-dimensional $K[HC/C]$-submodules, it follows that

$$\hat{E}_{HC/C} = \widehat{X_1} \oplus \cdots \oplus \widehat{X_t} ,$$

where $\widehat{X_i} = \operatorname{Irr}(X_i)$ is a 1-dimensional irreducible $K[HC/C]$-module. By Theorem 3.1 above, the claim is proved.

Now, let $\lambda_i \in \operatorname{Irr}(E_i)$ such that $I_G(\lambda_i) = C_G(E_i)$, and let

$$\lambda = \lambda_1 \times \cdots \times \lambda_t .$$

Then

$$I_G(\lambda) = \bigcap_{i=1,\dots,s} C_G(E_i) = C_G(F) = F.$$

Thus $\lambda^G \in \operatorname{Irr}(G)$, λ^G has p-defect zero and the proof is complete.

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