PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 4, APRIL 1992

AN ARC OF FINITE 2-MEASURE THAT IS NOT RATIONALLY CONVEX

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. A subset of $\mathbf{C}^2$ having the properties in the title is constructed.

If K is a compact subset of $\mathbf{C}^n$, let $\pi(K)$ and $\rho(K)$ denote the *polynomially convex hull* of K and the *rationally convex hull* of K, respectively. We then have $\pi(K) \supset \rho(K) \supset K$. If K has zero two-dimensional Hausdorff measure, it is well known that $\rho(K) = K$. If Γ is a compact Jordan arc (homeomorph of $[0, 1]$) with finite one-dimensional Hausdorff measure, a deep result of Alexander [A] states that $\pi(\Gamma) = \Gamma$. Thus Gamelin [G] has asked the following questions:

(1) If Γ is a compact Jordan arc in $\mathbf{C}^n$ that has finite two-dimensional Hausdorff measure, does it follow that $\rho(\Gamma) = \Gamma$?
(2) If $\alpha \in (1, 2]$ and Γ is a compact Jordan arc in $\mathbf{C}^n$ that has finite α-dimensional Hausdorff measure, does it follow that $\pi(\Gamma) = \Gamma$?

The purpose of this note is to point out that question 1 can be answered by combining a recent theorem of Uy [U] with a well-known construction of Wermer [W] and Rudin [R]. This also gives an answer to question 2 in case $\alpha = 2$.

Theorem. *If $n \geq 2$, there exists a compact Jordan arc $\Gamma \subset \mathbf{C}^n$ that has finite two-dimensional Hausdorff measure, but $\rho(\Gamma) \neq \Gamma$. (In particular, $\pi(\Gamma) \neq \Gamma$.)*

Proof. We may assume $n = 2$. Let E be a totally disconnected compact subset of $\mathbf{C}$ with positive two-dimensional Lebesgue measure. From the theorem of Uy [U] there exists a bounded Lipschitz function $f: \mathbf{C} \to \mathbf{C}$ such that the restriction $f|_{\mathbf{C}\setminus E}$ is holomorphic and nonconstant, and $f(\infty) = 0$. We define $M: \overline{\mathbf{C}} \to \mathbf{C}^2$ by $M(z) \equiv (f(z), zf(z))$, and note that M is injective on the complement of $f^{-1}(0)$. Then the image $X = M(E)$ is totally disconnected (in fact, if Y denotes the quotient space obtained from E by identifying all points of $E \cap f^{-1}(0)$, then Y is totally disconnected and M induces a homeomorphism from Y onto X.) It then follows from a theorem of Antoine [An] that there is a Jordan arc $\Gamma \subset \mathbf{C}^2$ containing X such that $\Gamma\setminus X$ is locally polygonal. This property, and the fact that M is Lipschitz on E, show that the set $\Gamma =$

Received by the editors October 23, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32E30; Secondary 32E20.
Research was supported by NSERC of Canada and FCAR du Québec.

$(\Gamma\backslash X) \cup M(E)$ has finite two-dimensional Hausdorff measure. Now using the fact that the restriction of M to $W = \mathbf{C}\backslash[E \cup f^{-1}(0)]$ is an immersion, while the set $\Gamma\backslash X$ is locally rectifiable, we see that the set $M(W)\backslash(\Gamma\backslash X) = M(W)\backslash\Gamma$ is nonempty; we select a point $\omega \in W$ such that $M(\omega) \notin \Gamma$.

To complete the proof of the theorem, it suffices to show that $M(\omega) \in \rho(\Gamma)$. If not, it is well known that there is a polynomial p on $\mathbf{C}^2$ that vanishes at $M(\omega)$ but never on Γ. Then $g \equiv p \circ M : \overline{\mathbf{C}} \to \mathbf{C}$ is continuous and $g|_{\overline{\mathbf{C}}\backslash E}$ is holomorphic. Moreover, g never vanishes on E, so g vanishes in only finite many points of $\mathbf{C}\backslash E$; let these points be denoted (according to multiplicity) by $\zeta_1 = \omega$, $\zeta_2, \dots, \zeta_r$, where $r \geq 1$. Then $h(z) \equiv g(z)/(z-\zeta_1)\cdots(z-\zeta_r)$ never vanishes on $\mathbf{C}$, so there exists a continuous function $H : \mathbf{C} \to \mathbf{C}$ such that $h \equiv e^H$ on $\mathbf{C}$. We have now obtained a deleted neighborhood of infinity, $\mathbf{C}\backslash E$, on which there are holomorphic functions h and H, such that $h \equiv e^H$ and $h(\infty) = 0$; this is impossible, so the theorem is proved.

References

[A] H. Alexander, *Polynomial approximation and hulls in sets of finite linear measure in* $\mathbf{C}^n$, Amer. J. Math. **93** (1971), 65–74.

[An] L. Antoine, *Sur la possibilité d'étendre l'homéomorphie de deux figures à leur voisinage*, C. R. Acad. Sci. Paris Sér. I Math. **171** (1920), 661–663.

[G] T. W. Gamelin, *Polynomial approximation on thin sets* (Symposium on Several Complex Variables, 1970), Lecture Notes in Math., vol. 184, Springer-Verlag, Berlin, 1970, pp. 50–78.

[R] W. Rudin, *Subalgebras of spaces of continuous functions*, Proc. Amer. Math. Soc. **7** (1956), 825–830.

[U] Nguyen Xuan Uy, *Removable sets of analytic functions satisfying a Lipschitz condition*, Ark. Mat. **17** (1979), 19–27.

[W] J. Wermer, *Polynomial approximation on an arc in* $\mathbf{C}^3$, Ann. of Math. (2) **62** (1955), 269–270.

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