

MEASURABLE LINEAR FUNCTIONALS AND OPERATORS ON FRÉCHET SPACES

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ABSTRACT. The structure of measurable linear functionals and operators on Fréchet spaces with so-called stochastic bases is described.

1. INTRODUCTION

In this paper we study measurable linear functionals and operators on Fréchet spaces. Measurable linear functionals have been studied by many authors (see e.g., [1, 4, 6, 10, 11]), however, measurable linear operators have been seldom treated. The theory of measurable linear operators on Hilbert spaces was presented in [10] and on Banach spaces in [12].

The purpose of this paper is to investigate the structure of measurable linear functionals and operators on Fréchet spaces with so-called stochastic bases. The notion of the stochastic basis was introduced by Herer in [3]. In this paper we will admit a slight modification of this concept.

Our results extend and generalize the corresponding facts that have been proved for particular cases (see [2, 4, 8]). However, all these cases, until now, have been treated separately. In this paper they will appear as the corollaries of our results. Moreover, we shall present a lot of new similar examples.

Our results were announced previously in [13] without proofs.

2. PRELIMINARIES

Let X and Y be real separable Fréchet spaces (i.e., complete locally convex linear metric spaces), and let μ be a Borel probability measure on X . Denote by $\mathcal{B}(X)$ the Borel σ -algebra on X and by $\mathcal{B}_\mu(X)$ the completion in measure μ of $\mathcal{B}(X)$.

An operator A defined on a linear subset $D_A \subset X$ with values in Y is called a μ -measurable linear operator if

- (a) $D_A \in \mathcal{B}_\mu(X)$ and $\mu(D_A) = 1$;
- (b) A is a measurable mapping with respect to $(\mathcal{B}_\mu(X), \mathcal{B}(Y))$;
- (c) A is linear on D_A .

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If $Y = R$ then we will say simply functional instead of operator.

Denote by X^* the topological dual of X , and let $\{x_n, f_n\}$ be a biorthogonal system in (X, X^*) , i.e., $f_i(x_j) = \delta_{ij}$ for $i, j = 1, 2, \dots$. Such a system is called a *stochastic basis* in (X, μ) if

- (a) there exists a Borel subset $T_\mu \subset X$ with $\mu(T_\mu) = 1$ such that for every $x \in T_\mu$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f_k(x)x_k = x;$$

- (b) functionals $\{f_n\}$ treated as random variables on (X, μ) are stochastically independent.

Therefore, if $\{x_n, f_n\}$ is a stochastic basis in (X, μ) then we can write that

$$x = \sum_{n=1}^{\infty} f_n(x)x_n$$

for every $x \in T_\mu$.

It is evident that T_μ is a linear subset of X .

The measure μ will be called nondegenerate if the measure of every proper closed linear subspace in X is less than 1.

We will say that the measure μ is symmetric if $\mu(B) = \mu(-B)$ for every $B \in \mathcal{B}(X)$. By μ^s we shall denote the symmetrization of μ , that is, a symmetric measure defined by the formula $\mu^s(B) = \frac{1}{2}(\mu(B) + \mu(-B))$ for $B \in \mathcal{B}(X)$.

3. MAIN RESULTS

Let X be a separable Fréchet space, and let μ be a Borel probability measure on X . Suppose now that there exists a stochastic basis $\{x_n, f_n\}$ in (X, μ) .

Proposition 1. *Assume that the measure μ is nondegenerate. If L is a μ -measurable linear subset of X such that $\mu(L) = 1$ then $x_n \in L$ for every $n = 1, 2, \dots$.*

Proof. Suppose in the first place that the measure μ is symmetric. Denote by S the set of linear transformations $s: T_\mu \rightarrow T_\mu$ such that $s(x_n) = \varepsilon_n x_n$ where $\varepsilon_n = \pm 1$ and $\varepsilon_n = 1$ for all $n \geq n_0$ for some $n_0 \geq 1$. Therefore if $x \in T_\mu$ and $x = \sum_n f_n(x)x_n$ then $s(x) = \sum_n \varepsilon_n f_n(x)x_n$. S is obviously a countable set. Since the measure μ is symmetric and functionals $\{f_n\}$ are independent as random variables on (X, μ) , we have that

$$(1) \quad \mu(s(B)) = \mu(B)$$

for all $B \in \mathcal{B}(X)$, ($B \subset T_\mu$), and every $s \in S$.

Since μ is a Radon measure on X , we can find a σ -compact linear set $L_0 \subset L$ such that $\mu(L_0) = \mu(L) = 1$. So we may as well assume that L itself is a σ -compact and consequently a Borel subset of X .

Let $L_1 = L \cap T_\mu$. Then L_1 is a linear Borel set and $\mu(L_1) = 1$. Let us put

$$L_0 = \bigcap_{s \in S} s(L_1).$$

L_0 is a linear Borel set and from (1) it follows that $\mu(L_0) = 1$.

Now let $n \geq 1$ be fixed. Then there is $y \in L_0$ such that $f_n(y) \neq 0$. In fact, if for every $y \in L_0$, $f_n(y) = 0$, then $\mu\{y \in X : f_n(y) = 0\} = 1$, which is impossible because the measure μ is nondegenerate.

Let $s \in S$ be the transformation with $\varepsilon_n = -1$ and $\varepsilon_k = 1$ for $k \neq n$. Since $s(L_0) = L_0$ then $s(y) \in L_0$. Let us remark that $y - s(y) = 2f_n(y)x_n$. Hence $x_n = (2f_n(y))^{-1}(y - s(y))$. Moreover, since L_0 is a linear set and $y, s(y) \in L_0$, we have $x_n \in L_0$. But $L_0 \subset L_1$ whence $x_n \in L_1$ and consequently $x_n \in L$. This proves our fact in the case when μ is a symmetric measure.

If now μ is any measure then we may consider the symmetrization μ^s of μ . Since T_μ is a linear set and $\mu(T_\mu) = 1, \mu^s(T_\mu) = 1$. Therefore $\{x_n, f_n\}$ is also a stochastic basis for μ^s . Moreover, since μ is a nondegenerate measure, so is μ^s . Thus if L is a μ -measurable linear set such that $\mu(L) = 1$ then also $\mu^s(L) = 1$ and by virtue of the first part of this proof, we have that $x_n \in L$ for every $n = 1, 2, \dots$. This completes the proof of our assertion.

Now let

$$C(\mu) = \left\{ (a_n) \in R^\infty : \sum_{n=1}^\infty a_n f_n(x) < \infty \mu\text{-almost everywhere} \right\}.$$

It is evident that if $(a_n) \in C(\mu)$ then the formula

$$(2) \quad f(x) = \sum_{n=1}^\infty a_n f_n(x)$$

defines a μ -measurable linear functional on X . Our aim will be to show that for the symmetric and nondegenerate measure μ , every μ -measurable linear functional on X has a representation in the form (2).

Let us before remark that if f is a μ -measurable linear functional defined on D_f and if we suppose that the measure μ is nondegenerate, then by Proposition 1 we have $x_n \in D_f$ for all $n = 1, 2, \dots$. Therefore every μ -measurable linear functional is in this case defined for each x_n ($n = 1, 2, \dots$). The after-mentioned fact shows that the values of the functional f on vectors x_n ($n = 1, 2, \dots$) uniquely determine this functional.

Proposition 2. *Suppose that the measure μ is nondegenerated, and let f be a μ -measurable linear functional on X (defined on D_f). If $f(x_n) = 0$ for $n = 1, 2, \dots$ then $f = 0$ μ -a.e.*

Proof. For every $x \in D_f \cap T_\mu$ and for all $n = 1, 2, \dots$, we have

$$x = \sum_{k=1}^n f_k(x)x_k + \sum_{k=n+1}^\infty f_k(x)x_k.$$

Hence

$$f(x) = \sum_{k=1}^n f_k(x)f(x_k) + f\left(\sum_{k=n+1}^\infty f_k(x)x_k\right) = f\left(\sum_{k=n+1}^\infty f_k(x)x_k\right).$$

Thus f depends only on the random variables $\{f_k : k \geq n + 1\}$ for every $n = 1, 2, \dots$. But the random variables $\{f_n\}$ are independent on (X, μ) . Hence by virtue of the Kolmogorov's 0-1 law, we obtain that there is $c \in R$

such that $f = c$ μ -a.e. Since the functional f is linear then $c = 0$. Therefore $f = 0$ μ -a.e.

Corollary. *Suppose that the measure μ is nondegenerate. If f and g are μ -measurable linear functionals on X and $f(x_n) = g(x_n)$ for $n = 1, 2, \dots$, then $f = g$ μ -a.e.*

The main results of this paper are the following two theorems.

Theorem 1. *Assume that the measure μ is symmetric and nondegenerate. If f is a μ -measurable linear functional on X then*

- (a) $\{f(x_n)\}_{n=1}^{\infty} \in C(\mu)$;
- (b) $f(x) = \sum_{n=1}^{\infty} f_n(x)f(x_n)$ μ -a.e.;
- (c) the representation (b) is unique.

Proof. (a) Let $x \in D_f \cap T_{\mu}$ and $n \in N$. Then

$$x = \sum_{k=1}^n f_k(x)x_k + x(n), \quad \text{where } x(n) = \sum_{k=n+1}^{\infty} f_k(x)x_k.$$

Hence

$$(3) \quad f(x) = \sum_{k=1}^n f_k(x)f(x_k) + f(x(n)).$$

Define $a_n = f(x_n)$, $X_n(x) = a_n f_n(x)$, and $Y_n(x) = f(x(n))$. Then X_1, X_2, \dots is a sequence of independent and symmetric random variables on (X, μ) . Also for every $n \in N$ the random variables X_1, X_2, \dots, Y_n are independent.

The equality (3) can be rewritten in the form

$$(4) \quad f(x) = \sum_{k=1}^n X_k(x) + Y_n(x).$$

Denote by φ , φ_n , and ψ_n the characteristic functions of f , X_n , and Y_n . Then from (4) we have

$$\varphi(t) = \psi_n(t) \prod_{k=1}^n \varphi_k(t).$$

Hence

$$(5) \quad |\varphi(t)| \leq \lim_{n \rightarrow \infty} \prod_{k=1}^n |\varphi_k(t)|.$$

Since X_1, X_2, \dots are independent random variables, the series $\sum_{n=1}^{\infty} X_n$ is either convergent μ -a.e. or divergent μ -a.e. If this series is divergent then by [7, Corollary 2, p. 251, Theorem b, p. 250] we have that $\lim_{n \rightarrow \infty} \prod_{k=1}^n |\varphi_k(t)| = 0$. Then from (5) it follows that $\varphi(t) = 0$ for $t \neq 0$. But this is impossible because φ is a characteristic function (of the random variable f). Therefore $\sum_{n=1}^{\infty} X_n < \infty$ μ -a.e., that is, $\sum_{n=1}^{\infty} f(x_n)f_n(x) < \infty$ μ -a.e. This means that $\{f(x_n)\}_{n=1}^{\infty} \in C(\mu)$.

(b) Let $g(x) = \sum_{n=1}^{\infty} f_n(x)f(x_n)$. Since by (a) $\{f(x_n)\} \in C(\mu)$, g is a μ -measurable linear functional on X . But $f(x_n) = g(x_n)$ for every $n \in N$

N . Hence by the Corollary we have that $f = g$ μ -a.e., that is, $f(x) = \sum_n f_n(x)f(x_n)$ μ -a.e.

(c) Suppose that the functional f has another representation in the form (b), i.e., let $f(x) = \sum_n a_n f_n(x)$ μ -a.e. But then $f(x_n) = a_n$ for $n = 1, 2, \dots$. Thus the representation (b) is really unique.

Theorem 2. *Let the measure μ be symmetric and nondegenerate, and let Y be a separable Banach space. If A is a μ -measurable linear operator from X into Y then A may be uniquely represented in the form*

$$(6) \quad Ax = \sum_{n=1}^{\infty} f_n(x)Ax_n \quad \mu\text{-a.e.}$$

Proof. For every $f \in Y^*$ the formula $f_A(x) = f(Ax)$ defines a μ -measurable linear functional on X . Therefore by Theorem 1 we obtain

$$f_A(x) = \sum_{n=1}^{\infty} f_n(x)f_A(x_n) \quad \mu\text{-a.e.}$$

or

$$(7) \quad f(Ax) = \sum_{n=1}^{\infty} f_n(x)f(Ax_n) \quad \mu\text{-a.e.}$$

for every $f \in Y^*$.

Regarding A as a random vector with values in Y , we have $\{f_n(x)Ax_n\}_{n=1}^{\infty}$ is a sequence of independent and symmetric random vectors and by Theorem Ito-Nisio [5] we obtain (6). The uniqueness of (6) also follows from Theorem 1.

4. EXAMPLES

Now we will present some examples of measures for which there exist stochastic bases and consequently Theorems 1 and 2 about the structure of measurable linear functionals and operators hold. In any case we also want to give some information about the set $C(\mu)$, that characterizes the space of μ -measurable linear functionals.

Example 1. Let X be a separable Fréchet space and μ a symmetric Gaussian measure on X . Then there is a stochastic basis in (X, μ) (see [3]). It is easy to show that in this case $C(\mu) = l_2$.

For the particular case of the Gaussian measure on a separable Banach space Theorem 1 has been obtained in [8].

Example 2. Let X be a separable Banach space and μ a symmetric p -stable measure with discrete spectral measure ($0 < p < 2$). This case was considered in [2] where the construction of a stochastic basis for such measure μ was given. It was also shown that in this case $C(\mu) = l_p$.

Example 3. Let $X = R^{\infty}$ and $\mu = \prod_{n=1}^{\infty} \mu_n$ be a probability product measure on X with symmetric and nondegenerate factors. Measurable linear functionals for such measures μ have been considered in detail in [4]. But it is easy to see that this example is also a special case of our results. Indeed, if e_n is the

n th unit vector in R^∞ and $f_n(x) = x_n$ for $x = (x_n) \in R^\infty$ then the system $\{e_n, f_n\}$ is a stochastic basis in (R^∞, μ) .

The set $C(\mu)$ depends in this case on measures μ_n . If, for example, all measures μ_n are identical then $C(\mu) \subset l_2$ (see [4]). For other information about the set $C(\mu)$ in this case we refer to paper [4].

Example 4. Let X be a separable Fréchet space and μ a symmetric measure of second order on X (i.e., $\int_X \|x\|^2 d\mu(x) < \infty$ for every continuous seminorm $\|\cdot\|$ in X).

Measurable linear functionals and operators for such measures have not been considered before. But in this case Theorems 1 and 2 also hold if we suppose that there exist sufficiently rich stochastically independent continuous linear functions on X . Indeed if μ is a symmetric measure of second order on X and if we suppose that there exists a sequence $\{g_n\} \subset X^*$ such that $\{g_n\}$ are stochastically independent random variables on (X, μ) and that $\{g_n\}$ is total in X^* with respect to the $L^2(\mu)$ -metric, then there exists a stochastic basis $\{x_n, f_n\}$ in (X, μ) (see [9]).

Now we show that in this case $C(\mu) \supset l_2$.

Let us remark that without loss of generality (by [9, Theorem 2]) we can assume that X is a separable Banach space (with the norm $\|\cdot\|$). We may also suppose that $\|f_n\| = 1$ for $n = 1, 2, \dots$.

To prove that $l_2 \subset C(\mu)$ let us assume that $(a_n) \in l_2$ and put $X_n(x) = a_n f_n(x)$ ($n = 1, 2, \dots$). Then $\{X_n\}$ is a sequence of independent random variables on (X, μ) . Moreover for every $n \in N$ we have

$$EX_n^2 = \int_X (a_n f_n(x))^2 \mu(dx) \leq a_n^2 \int_X \|x\|^2 \mu(dx).$$

Hence

$$(8) \quad E \left(\sum_{j=n+1}^{n+m} X_j^2 \right) \leq \sum_{j=n+1}^{n+m} EX_j^2 \leq \int_X \|x\|^2 \mu(dx) \cdot \sum_{j=n+1}^{n+m} a_j^2.$$

Since μ is a measure of second order, $\int_X \|x\|^2 \mu(dx) < \infty$. Hence, taking into account the assumption (i.e., that $(a_n) \in l_2$), we infer from (8) that the series $\sum_n X_n$ is convergent in $L_2(\mu)$ and consequently also in probability μ . But the random variables $\{X_n\}$ are independent, therefore from the convergence of the series $\sum_n X_n$ in probability μ follows the convergence of this series μ -almost everywhere (see [5, Theorem 3.1]). Thus $\sum_n X_n < \infty$ μ -a.e., that is, $\sum_n a_n f_n(x) < \infty$ μ -a.e. This means that $(a_n) \in C(\mu)$. Our statement is proved.

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