PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 1, May 1992

APPROXIMATION OF CONVEX BODIES BY TRIANGLES

MAREK LASSAK

(Communicated by William J. Davis)

ABSTRACT. We show that for every plane convex body C there exist a triangle T_1 and its image T_2 under a homothety with ratio $\frac{5}{2}$ such that $T_1 \subset C \subset T_2$. We prove the conjecture of Grünbaum that if C is centrally symmetric, then T_1, T_2 can be chosen so that their centroids coincide with the center of C.

Let C be a convex body in the Euclidean plane E^2. By the *centroid* of C we understand the center of mass of C. Of course, if C is centrally symmetric, the centroid is the center of symmetry of C. Denote by λC the homothetic copy of C with homothety center at the centroid of C and homothety ratio λ. By $w(C, l)$ we denote the width of C in the direction l.

By an *affine-regular hexagon* we mean an affine image of the regular hexagon. Besicovitch [2] proved that an affine-regular hexagon can be inscribed in every plane convex body.

Lemma. *Let $M \subset E^2$ be a centrally symmetric convex body. We can inscribe in M an affine-regular hexagon H whose center is the center of symmetry of M such that $M \subset \frac{3}{2}H$.*

Proof. Lemma 1 of [5] says that for every direction l there is exactly one affine-regular hexagon $H(l) = abcdef$ (with vertices a, b, c, d, e, f in the positive order) inscribed in M such that the vector $\overrightarrow{ac}$ has direction l, and that the vertices of $H(l)$ vary continuously along the boundary of M as l rotates. We define the direction l^* associated with l as the direction of the vector $\overrightarrow{ap}$, where p is the perpendicular projection of a on the straight line through c and d. It is clear that l^* is defined for every direction l and that l^* depends continuously on l.

Since M is centrally symmetric, the center of symmetry of $H(l)$ is always in the center o of symmetry of M (see the proof of Lemma 1 of [5]).

Denote by m the direction of $\overrightarrow{bd}$. Of course, $H(l) = H(m)$. There is a supporting line of M through d. The intersection of this line with the line containing the segment bc is denoted by v, and with the line containing ef by z (see Figure 1). Let Q denote the convex cone with vertex d and bounding lines through c and e. Since $H(l)$ is inscribed in M, the set $M \setminus Q$ is contained in the union of triangles cdv and dez. Moreover, an argument

Received by the editors February 22, 1990 and, in revised form, June 18, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 52A10.

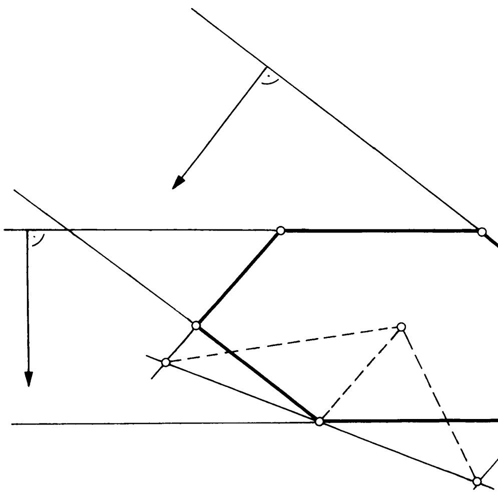

FIGURE 1

from plane geometry establishes that

$$\text{if } w(ov, l^*) \geq \tfrac{3}{2}w(od, l^*), \text{ then } w(oz, m^*) \leq \tfrac{3}{2}w(od, m^*).$$

Both the facts and the central symmetry of the construction imply

$$\text{(1)} \qquad \text{if } w(M, l^*) \geq \tfrac{3}{2}w(H(l), l^*), \text{ then } w(M, m^*) \leq \tfrac{3}{2}w(H(l), m^*).$$

Consider the case when there exists a direction l_0 such that

$$\text{(2)} \qquad w(M, l_0^*) > \tfrac{3}{2}w(H(l_0), l_0^*);$$

in the opposite case Lemma is trivially true.

Denote by m_0 the direction of $\overrightarrow{b_0d_0}$, where $H(l_0) = a_0b_0c_0d_0e_0f_0$. By (1), (2), and by $H(l_0) = H(m_0)$ we have

$$\text{(3)} \qquad w(M, m_0^*) \leq \tfrac{3}{2}w(H(m_0), m_0^*).$$

Since the width of a convex body depends continuously on the direction (see [3, p. 78]) and since l^* continuously depends upon l, we see that $w(M, l^*)$ is a continuous function of l. Thus from (2) and (3), and from the fact that the vertices of $H(l)$ vary continuously as l rotates, we conclude that there is a direction l_2 such that

$$\text{(4)} \qquad w(M, l_2^*) = \tfrac{3}{2}w(H(l_2), l_2^*).$$

Let $H(l_2) = a_2b_2c_2d_2e_2f_2$. Denote by l_1 and l_3 the directions of the vectors $\overrightarrow{b_2d_2}$ and $\overrightarrow{f_2b_2}$, respectively. Observe that (1) remains true for m being the direction of $\overrightarrow{fb}$. From both versions of (1) and from (4) we obtain

$$w(M, l_i^*) \leq \tfrac{3}{2}w(H(l_2), l_i^*)$$

for $i = 1$ and $i = 3$. This and (4) imply $M \subset \tfrac{3}{2}H(l_2)$.

The proof is complete.

As observed by a referee, our Lemma is stronger than Theorem 2 of Asplund [1] that $M \subset \tfrac{3}{2}H$ for an affine-regular hexagon $H \subset M$. We do not see any

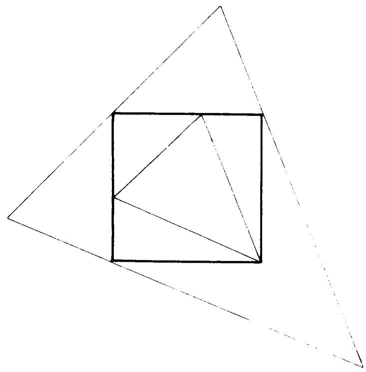

FIGURE 2

way to prove our Theorem directly from Asplund's result; the fact that H is inscribed in M is essential in the proof. By the way, the referee noticed that Asplund proved a stronger result than he stated. His proof indicates that H can be chosen in such a way that $\frac{3}{2}H$ is circumscribed about M.

We are ready to confirm one of the conjectures of Grünbaum presented in [4, p. 260].

Theorem. *Let $M \subset E^2$ be a centrally symmetric convex body. We can inscribe in M a triangle T whose centroid is the center of symmetry of M such that $M \subset \frac{5}{2}T$.*

Proof. By Lemma it is possible to inscribe in M an affine-regular hexagon $H = abcdef$ such that $M \subset \frac{3}{2}H$. Three of the lines containing the sides of H bound a triangle S_1 containing H and the other three a triangle S_2. Since H is inscribed in M, we see that $M \subset S_1 \cup S_2$.

Of course, the triangle T with the vertices a, c, e is inscribed in M.

Denote by o the center of symmetry of M. It is easy to see that o is the center of H (see the beginning of the proof of Lemma 1 of [5]). Thus o is the centroid of T. Consequently, from the inclusions $M \subset \frac{3}{2}H$ and $M \subset S_1 \cup S_2$ we obtain $M \subset \frac{5}{2}T$, which ends the proof.

The example of M being a parallelogram shows that the ratio $\frac{5}{2}$ in the theorem cannot be lessened (see [4, p. 259]).

In connection with Theorem, the author conjectures that for every convex body $C \subset E^2$ there is a triangle T inscribed in C such that $C \subset (1 + \frac{3}{5}\sqrt{5})T$. The ratio $1 + \frac{3}{5}\sqrt{5}$ cannot be improved because of the example of a parallelogram as C (see Figure 2).

We conjecture that for every convex body $C \subset E^2$ there is a triangle T_1 and a homothetic copy T_2 of T_1, where the ratio of the homothety is $\cos^2 36°/\sin 18° = 2.118\ldots$, such that $T_1 \subset C \subset T_2$. An easy but time consuming calculation shows that the above ratio cannot be lessened when C is the regular pentagon (see Figure 3 on next page). Recall that if C is centrally symmetric, the ratio is 2 (see [4, p. 259]). Here is an estimate for an arbitrary convex body $C \subset E^2$.

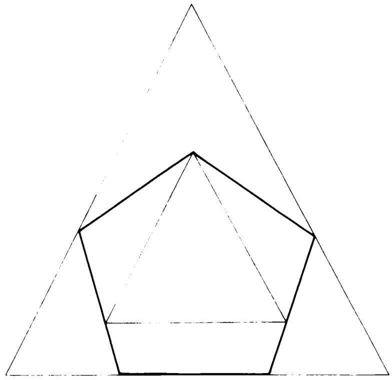

FIGURE 3

Proposition. *For every convex body $C \subset E^2$ there are a triangle T_1 and its image T_2 under a homothety with the ratio $\frac{5}{2}$ such that $T_1 \subset C \subset T_2$.*

Proof. Compactness arguments show that we can inscribe in C a triangle T_1 of maximal area. Let T_2 be the homothetic copy of T_1 containing C with the smallest possible positive ratio μ of the homothety. Of course, there are points a, b, c of C on different sides of T_2. Let $S = (-2)T_1$.

If we suppose that a point $p \in C$ is not in S, then a triangle whose vertices are p and two of the vertices of T_1 is contained in C and has the area greater than the area of T_1. This contradicts the choice of T_1 and shows that $C \subset S$.

Of course, S is the copy of T_2 under a homothety with the negative ratio $-2/\mu$. Suppose that $\mu > \frac{5}{2}$. Then $-2/\mu > -\frac{4}{5}$. Thus from $a, b, c \in C \subset S$ we deduce that a, b, c are not contained in the three triangles that are homothetic copies of T_2 with homothety centers at the vertices of T_2 and ratio $\frac{1}{5}$. As a consequence, the area of the triangle abc is greater than $\frac{4}{25}$ of the area of T_2; i.e., greater than the area of T_1. This contradiction with the maximality of the area of T_1 shows that $\mu \leq \frac{5}{2}$.

REFERENCES

1. E. Asplund, *Comparison between plane symmetric convex bodies and parallelograms*, Math. Scand. **8** (1960), 171–180.
2. A. S. Besicovitch, *Measure of assymetry of convex curves*, J. London Math. Soc. **23** (1945), 237–240.
3. H. G. Eggleston, *Convexity*, Cambridge Univ. Press, Cambridge, 1969.
4. B. Grünbaum, *Measures of symmetry of convex sets*, Convexity, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc., Providence, R. I., 1963, pp. 233–270.
5. M. Lassak, *Approximation of convex bodies by centrally symmetric bodies*, J. London Math. Soc. (2) **40** (1989), 369–377.

INSTYTUT MATEMATYKI I FIZYKI ATR, 85-790 BYDGOSZCZ, POLAND