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A TEST THEOREM ON COHERENT GCD DOMAINS

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ABSTRACT. Let R be a commutative indecomposable coherent ring. Then the following statements are equivalent: (i) R is a GCD domain; (ii) R_M is a GCD domain for every maximal ideal of M of R, and every finitely generated projective ideal in R is principal; (iii) every two-generated ideal in R has finite projective dimension, and every finitely generated projective ideal in R is principal. Auslander-Buchsbaum's Theorem, etc. can be obtained from the result above.

Let R be a commutative ring with $1 \neq 0$ in this paper. R is called a GCD domain if every two elements $a, b \in R$ have a greatest common divisor (denoted by $[a, b]$) in R. R is said to be indecomposable if 1 is the only nonzero idempotent of R. A coherent ring is a ring whose finitely generated ideals are finitely presented.

Definition. We say that R has PPC if every finitely generated projective ideal in R is a principal ideal.

Theorem. *Let R be an indecomposable coherent ring. Then the following statements are equivalent*:

(i) *R is a GCD domain.*
(ii) *R_M is a GCD domain for every maximal ideal M of R, and R has PPC.*
(iii) *Every two-generated ideal in R has finite projective dimension, and R has PPC.*

We first quote two results from Vasconcelos [1] that will be fundamental to this note.

Lemma 1 [1, Corollary 5.16]. *Let R be a coherent local ring such that every principal ideal has finite projective dimension. Then R is a domain.*

Lemma 2 [1, Corollary 5.20]. *Let R be a coherent domain, and let I be a finitely generated ideal of finite projective dimension. Then $I = d(R/I)J$, where $d(R/I)$ is an invertible ideal and J is an ideal of R satisfying $J^{-1} = R$.*

Proof of Theorem. (i) $\Rightarrow$ (ii) Let M be a maximal ideal of R. Let $\frac{a}{1}, \frac{b}{1} \in R$, and write $c = [a, b]$. if $\frac{d}{1}$ is a common divisor of $\frac{a}{1}$ and $\frac{b}{1}$ in R_M then $d|sa$

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and $d|tb$ for some $s, t \in R - M$. Thus d is a common divisor of sta and stb in R. Since $stc = [sta, stb]$, we have $d|stc$. But $st \notin M$. Hence $\frac{d}{1}|\frac{c}{1}$, that is, $\frac{c}{1} = [\frac{a}{1}, \frac{b}{1}]$. Therefore, R_M is a GCD domain for every maximal ideal M of R. Finally, let $I = (a_1, \dots, a_n)$ be a finitely generated projective ideal of R. Then I is an invertible ideal, that is, $II^{-1} = R$. It is straightforward to check that $I^{-1} = \frac{1}{c}R$. Hence $I = cR$. Therefore R has PPC.

(ii) $\Rightarrow$ (iii) Let $a, b \in R$. Consider the exact sequence

$$0 \longrightarrow (a) \cap (b) \longrightarrow R^2 \longrightarrow (a, b) \longrightarrow 0.$$

Let M be any maximal ideal of R. Since R_M is a GCD domain, $(a)_M \cap (b)_M$ is a principal ideal of R_M, and hence $(a)_M \cap (b)_M$ is 0 or R_M-free. It follows that $(a) \cap (b)$ is a flat ideal of R. But $(a) \cap (b)$ is finitely presented because R is a coherent ring. Hence $(a) \cap (b)$ is projective, which shows that $pd_R(a, b) \leq 1$. Therefore, R has PPC.

(iii) $\Rightarrow$ (i) We first prove that R is a domain. Let $0 \neq a \in R$. Consider the exact sequence

$$0 \longrightarrow \operatorname{ann}(a) \longrightarrow R \longrightarrow aR \longrightarrow 0.$$

Let M be any maximal ideal of R. By Lemma 1, R_M is a domain and hence aR_M is 0 or R_M-free. Now aR is a projective ideal of R, that is, $R = \operatorname{ann}(a) \oplus I$ for some ideal I of R. But R is indecomposable and $a \neq 0$. Hence $\operatorname{ann}(a) = 0$ and R is a domain.

Finally, let $a, b \in R$. We may assume that a, b are nonzero and nonunit. Write $I = (a, b)$. By Lemma 2, $(a, b) = d(R/I)(u, v)$, where $(u, v)^{-1} = R$. Since $d(R/I)$ is an invertible ideal, $d(R/I)$ is finitely generated projective, and hence $d(R/I) = cR$ for some $c \in R$. Thus $(a, b) = c(u, v)$ and $a = cu$, $b = cv$. It is clear that $c = [a, b]$. Therefore, R is a GCD domain.

A ring R is called regular if every finitely generated ideal in R has finite projective dimension.

Corollary 1 (Auslander-Buchsbaum's Theorem [2]). *A Noetherian regular local ring is a unique factorization domain.*

Corollary 2 (Vasconcelos's Theorem [1]). *A coherent regular local ring is a GCD domain.*

Corollary 3 (Kaplansky's Theorem [3]). *Let R be a Noetherian regular domain in which every invertible ideal is principal. Then R is a unqiue factorization domain.*

Proof. Noting that a local ring has PPC and that every projective ideal in a domain is invertible, we have Corollaries 1–3 by the theorem.

Corollary 4. *Let R be a coherent local ring (or an indecomposable semilocal ring). Then R is a GCD domain if and only if every two-generated ideal in R has finite projective dimension.*

Corollary 5. *Let R be an indecomposable coherent regular ring. Then R is a GCD if and only if R has PPC.*

Corollary 6. *Let R be an indecomposable coherent ring in which every two-generated ideal has finite projective dimension. Then R is an integrally closed domain.*

Proof. First, R is a domain (see the proof of (ii) $\Rightarrow$ (iii)). Second, R_M is a GCD domain for every maximal ideal M of R, and hence integrally closed. Therefore R is integrally closed.

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