PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 1, May 1992

κ-TOPOLOGIES FOR RIGHT TOPOLOGICAL SEMIGROUPS

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(Communicated by Johanthan M. Rosenberg)

ABSTRACT. Given a cardinal κ and a right topological semigroup S with topology τ, we consider the new topology obtained by declaring any intersection of at most κ members of τ to be open. Under appropriate hypotheses, we show that this process turns S into a topological semigroup. We also show that under these hypotheses the points of any subsemigroup T with $\operatorname{card} T \leq \kappa$ can be replaced by (new) open sets that algebraically behave like T. Examples are given to demonstrate the nontriviality of these results.

Let κ be a cardinal number. We call a κ-topology any topology for which the intersection of any family of open sets with no more than κ members is again open. Such topologies are easy to come by. If X is any topological space, the sets V of the form $V = \bigcap_{i\in I} U_i$ where $(U_i : i \in I)$ is any family of sets open in X with $\operatorname{card} I \leq \kappa$ provide a base of open sets for a κ-topology on X. We call this *the κ-topology on X*, we denote it by κ-X, we call its members *κ-open sets*, and we call κ-X the *κ-coreflection of X*.

A semigroup S with a completely regular topology is called *right topological* if all the maps $s \mapsto st$ are continuous for $t \in S$. The *topological center* of S is the subsemigroup

$$\Lambda(S) = \{s \in S : t \mapsto st \text{ is continuous}\}.$$

One of our main results is that if $\Lambda(S)$ contains a subset of cardinal κ that is dense in S then κ-S is a topological semigroup (that is, multiplication is jointly continuous). This theorem allows us to conclude that if $T \subseteq S$ is a subsemigroup, $\operatorname{card} T \leq \kappa$ and U is κ-open with $T \subseteq U$, then there is a κ-open semigroup T_0 with $T \subseteq T_0 \subseteq U$. These results hold in particular for Stone-Čech compactifications of discrete semigroups, the most important of which is $\beta\mathbb{N}$, where $\mathbb{N}$ is the semigroup of positive integers with addition. In the latter case we shall see that the semigroups T_0 are, in one sense, large.

In the terminology of [3, §2], a space with a κ-topology is a P_{κ^+}-space. When $\kappa = \aleph_0$ (as in the case of $\beta\mathbb{N}$), κ-topological spaces are more familiar as P-spaces (see [9, §1.65]). The space κ-X is then known as the P-space coreflection of X [9, Exercise 10B]). It is easy to see that in general κ-X is the *κ-coreflection*

Received by the editors August 1, 1990.
1991 *Mathematics Subject Classification*. Primary 22A15.
The second author gratefully acknowledges support received from the National Science Foundation via Grant DMS-8901058.

of X determined by the following categorical property: the κ-topology is the finest topology τ on X such that whenever Y has a κ-topology and $f: Y \to X$ is continuous then $f: Y \to (X, \tau)$ is continuous. In particular, every set open (or closed) in X is also κ-open (or κ-closed). The open sets in $\aleph_0$-X are precisely the unions of G_δ's in X. Obviously, if every point of a Hausdorff space X has a basis consisting of not more than κ neighborhoods then κ-X is discrete. We also remark that if κ is finite the κ-topology is just the original topology.

We shall restrict ourselves to completely regular (Hausdorff) spaces. For such spaces X and each infinite cardinal κ, every point has a base of open neighborhoods in κ-X that are closed sets in X (and so also closed in κ-X). For let $G = \bigcap_{i \in I} U_i$ be any basic open κ-neighborhood of x, with U_i open in X and $\operatorname{card} I \leq \kappa$. For each i, choose inductively a sequence (W_i^n) of neighborhoods of x open in X with $W_i^0 \subseteq U_i$ and $\operatorname{cl} W_i^{n+1} \subseteq W_i^n$ for $n \geq 0$. Then $\bigcap_{i,n} W_i^n = \bigcap_{i,n} \operatorname{cl} W_i^n$ is an open κ-neighborhood of x (since $\operatorname{card}(I \times \mathbb{N}) \leq \kappa$) contained in G and is obviously closed in X.

We now show that the κ-topology on a suitable right topological semigroup S has a strong continuity property.

Lemma 1. *Let S be a right topological semigroup and let κ be infinite. Suppose there is $K \subseteq \Lambda(S)$ such that K is dense in S and* $\operatorname{card} K \leq \kappa$. *Then multiplication is continuous from $S \times \kappa$-S to S.*

Proof. Take $a, b \in S$. Let U be any neighborhood of ab. Let U_0 be an open neighborhood of ab with $\operatorname{cl} U_0 \subseteq U$. Using the fact that S is right topological, find an open neighborhood V of a with $Vb \subset U_0$. Since $K \subseteq \Lambda(S)$, for each $k \in K \cap V$ we can find an open neighborhood W_k of b with $kW_k \subseteq U_0$. Then $G = \bigcap\{W_k : k \in K \cap V\}$ is a κ-open neighborhood of b. Since K is dense in S, it follows that $K \cap V$ is dense in V. Therefore if $v \in V$ and $g \in G$, we see that $vg \in \operatorname{cl}(K \cap V) \cdot g \subseteq \operatorname{cl} U_0 \subseteq U$, again using the fact that S is right topological. Thus $VG \subseteq U$, as required. □

Our remaining results are corollaries of Lemma 1.

Theorem 1. *Under the hypotheses of Lemma* 1, *κ-S is a topological semigroup.*

Proof. Let $a, b \in S$, and let E be a κ-neighborhood of ab, say $E = \bigcap_{i \in I} U_i$ with each U_i open in S and $\operatorname{card} I \leq \kappa$. For each $i \in I$ use Lemma 1 to find an open neighborhood V_i of a and a κ-neighborhood G_i of b with $V_i G_i \subseteq U_i$. Then $F = \bigcap_i V_i$, $G = \bigcap_i G_i$ are κ-neighborhoods of a, b respectively with $FG \subseteq E$. □

We can now prove our theorem about expanding semigroups. It says that, for subsemigroups T that are small enough, the points of T can be replaced by a family of κ-open sets that algebraically behave like T.

Theorem 2. *Let S be as in Lemma* 1. *Let $T \subseteq S$ be a subsemigroup with* $\operatorname{card} T \leq \kappa$. *Let E be a κ-open set with $T \subseteq E$. Then there is a disjoint family $\{T(t) : t \in T\}$ of closed κ-open subsets of S such that $t \in T(t)$ and $T(s)T(t) \subseteq T(st)$ for all $s, t \in T$.*

Proof. First we produce a disjoint family $\{E_0(t) : t \in T\}$ of κ-open sets with $t \in E_0(t) \subseteq E$ for each $t \in T$. For each pair s, t of distinct points of T

find disjoint open neighborhoods $U_t(s)$, $U_s(t)$ of s and t respectively. Then $E_0(t) = E \cap \bigcap\{U_s(t) : s \in T,\ s \neq t\}$ satisfies our requirements.

The proof is now by induction. For each $n > 0$ we find closed (in S) κ-open neighborhoods $E_n(t)$ of t for each $t \in T$. If $\{E_n(t) : t \in T\}$ has been determined, we use Theorem 1 to find κ-open neighborhoods $F^t_{n+1}(s)$, $G^s_{n+1}(t)$ with $F^t_{n+1}(s)G^s_{n+1}(t) \subseteq E_n(st)$; since there is a base for the κ-topology consisting of closed sets, we may (and do) presume that $F^t_{n+1}(s)$ and $G^s_{n+1}(t)$ are closed in S. Then $E_{n+1}(t) = \bigcap_{s\in T}(F^s_{n+1}(t) \cap G^s_{n+1}(t))$ is a closed κ-open set containing t, and the sets $\{E_{n+1}(t) : t \in T\}$ satisfy $E_{n+1}(s)E_{n+1}(t) \subseteq E_n(st)$ for all $s, t \in T$. Put $T(t) = \bigcap_{n=1}^{\infty} E_n(t)$. The family $\{T(t) : t \in T\}$ of closed, κ-open sets is disjoint and satisfies $T(s)T(t) \subseteq T(st)$ for all s, t. □

From Theorem 2 we see immediately that $T_0 = \bigcup_{t\in T} T(t)$ is a semigroup. If we put $\lambda = \operatorname{card} T$ then, being the union of λ closed sets, T_0 is λ-closed. This establishes the following corollary.

Corollary 1. *Let S, T, E be as in Theorem 2 and put $\lambda = \operatorname{card} T$. Then there is a λ-closed κ-open semigroup T_0 with $T \subseteq T_0 \subseteq E$.*

It is worth drawing attention to two special cases of Theorem 2.

Corollary 2. (i) *Let S be as in Lemma 1. If $e \in S$ is idempotent and E is a κ-neighborhood of e, there is a closed κ-open subsemigroup E_0 with $e \in E_0 \subseteq E$.*

(ii) *Let S be as in Lemma 1 and in addition compact. Let $T \subseteq S$ be a finite subsemigroup. Then there is a disjoint family $\{T(t) : t \in T\}$ of compact κ-open subsets of S with $t \in T(t)$ and $T(s)T(t) \subseteq T(st)$ for all $s, t \in T$.*

Corollary 2 (ii) was part of the original inspiration for this paper. It was discovered about 30 years ago that $\beta\mathbb{N}$ is naturally a compact right topological semigroup with an operation $+$ that extends addition in $\mathbb{N}$ and this semigroup has proved to be an invaluable tool in Ramsey Theory (see the surveys [6, 7]). It was clear from the beginning that $\mathbb{N} \subseteq \Lambda(\beta\mathbb{N})$ (and in fact $\mathbb{N} = \Lambda(\beta\mathbb{N})$, see [4]) and so the conclusions of Theorems 1 and 2 hold for $\beta\mathbb{N}$ with $\kappa = \aleph_0$. As remarked above, the $\aleph_0$-topology is the P-space topology.

Corollary 3. (i) *$\beta\mathbb{N}$ is jointly continuous in its P-space coreflection topology.*

(ii) *If T is a countable subsemigroup of $\beta\mathbb{N}$ and E is a G_δ with $T \subseteq E$ then there is a $G_{\delta\sigma}$ subsemigroup T_0 with $T \subseteq T_0 \subseteq E$. If T is finite, T_0 can be chosen to be a compact G_δ.*

In the case of $\beta\mathbb{N}$ we can add a little more. Each G_δ in $\beta\mathbb{N}$ is large in the sense that it contains a set open in the subspace $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ [9, Corollary 3.27]. Thus the semigroup T_0 is large in $\mathbb{N}^*$. In particular, if G is a countable semigroup in $\mathbb{N}^*$, then each element t of G can be 'expanded' to a compact set $G(t)$ with nonempty interior in $\mathbb{N}^*$ in such a way that $\{G(t) : t \in G\}$ is disjoint and satisfies $G(s) + G(t) \subseteq G(s+t)$ for all $s, t \in G$.

The question arises of whether we can arrange for $G(s) + G(t) = G(s+t)$ in this situation. The answer is that equality is never achieved. The reason is that $G(s) + G(t)$ is nowhere dense in $\mathbb{N}^*$ [4, Theorem 8.1] but $G(s+t)$ has nonempty interior.

If G is a finite group in $\beta\mathbb{N}$ then $G_0 = \bigcup_{t\in G} G(t)$ is compact subsemigroup with nonempty interior. However, whether $\beta\mathbb{N}$ contains nontrivial finite subgroups is unknown (and the question of the existence of such subgroups appears difficult).

The example $\beta\mathbb{N}$ shows that our results do have nontrivial content in at least one interesting case. Obtaining significant examples with $\kappa > \aleph_0$ is more difficult.

Example 1. *For each cardinal κ, there exist a compact right topological semigroup S and a subsemigroup T of S for which $T(t)$ is nontrivial for each $t \in T$.*

We begin with any infinite discrete semigroup L with $\operatorname{card} L = \kappa$. Put $S = \beta L$. Then S can be made a right topological semigroup with $L \subseteq \Lambda(S)$ (see [6] or [7]). Let U be the set of κ-uniform ultrafilters on L, that is $U = \{p \in S\colon \text{for all } A \subseteq S \text{ with } \operatorname{card} A < \kappa, p \notin \operatorname{cl}_S A\}$. By [3, Corollary 7.8(b)], $\operatorname{card} U = 2^{2^\kappa}$. By [3, Corollary 7.8(a)], if $p \in U$ then each neighborhood base of p has cardinal strictly greater than κ. So if T is a subsemigroup of S generated by a subset of U of cardinal κ, then for each $t \in T \cap U$ the κ-open set $T(t)$ consists of more than one point. Now [5, Theorem 2.5] gives conditions under which U is a semigroup, and this holds in particular if L is cancellative (in fact, U is then an ideal [5, Corollary 2.10]). Thus, for cancellative L, we have $T \subseteq U$ and our objective is achieved. □

For any semigroup S with a Hausdorff topology there is always a cardinal κ such that κ-S is jointly continuous, for when $\kappa = \operatorname{card} S$, κ-S is discrete. This suggests that we might use the smallest cardinal with this property as a measure of how discontinuous the multiplication of S is. Theorem 2 shows that sometimes a cardinal smaller than $\operatorname{card} S$ will do. We now give two examples, one to show that even for semigroups satisfying the conditions of Lemma 1, $\operatorname{card} S$ might be necessary. The other shows that a cardinal smaller than the κ of Theorem 1 is sometimes sufficient. (Of course, if S is jointly continuous to begin with, then $\kappa = 0$ is sufficient, but our example is not even separately continuous and is, we believe, more significant.)

Example 2. (i) *Given regular cardinal κ, there is a semigroup S with* $\operatorname{card} S = \kappa$ *that satisfies the conditions of Lemma* 1 *but for which λ-S is jointly continuous only if $\lambda \geq \kappa$.*

Let κ be a regular cardinal (which we regard as an ordinal), so that $\kappa = \operatorname{cf} \kappa$. Write the elements of $\bigoplus_\kappa \mathbb{Z}$, the direct sum of κ copies of $\mathbb{Z}$, as (transfinite) sequences $(z_\alpha)_{\alpha<\kappa}$ with $z_\alpha = 0$ for all but finitely many α. Define a total order on $\bigoplus_\kappa \mathbb{Z}$ by $(z_\alpha) < (w_\alpha)$ if and only if $z_\mu < w_\mu$ where $\mu = \max\{\alpha\colon z_\alpha \neq w_\alpha\}$ (this is a 'reverse' lexicographic order). Then $\bigoplus_\kappa \mathbb{Z}$ is a totally ordered group (with the usual operation $+$). We obtain S by adjoining to $\bigoplus_\kappa \mathbb{Z}$ two further elements ∞ and $-\infty$. We extend the order to S by writing $-\infty < x < \infty$ for all $x \in \bigoplus_\kappa \mathbb{Z}$, and we extend $+$ by writing $(-\infty) + x = -\infty$ and $\infty + x = \infty$ for all $x \in \bigoplus_\kappa \mathbb{Z}$, and $x + (-\infty) = -\infty$, and $x + \infty = \infty$ for all $x \in S$. We give S a topology by declaring each $x \in \bigoplus_\kappa \mathbb{Z}$ to be an isolated point and taking the intervals $[-\infty, x)$ to be basic neighborhoods of $-\infty$ and $(x, \infty]$ to be basic neighborhoods of ∞ for $x \in \bigoplus_\kappa \mathbb{Z}$. Then S is a right topological semigroup and $\Lambda(S) = \bigoplus_\kappa \mathbb{Z}$ (if $x \searrow -\infty$ then $\infty + x = \infty \not\to -\infty = \infty + (-\infty)$, so that $\infty \notin \Lambda(S)$; the other properties of S are equally easy to see).

Now if $\lambda < \kappa$ $(= \operatorname{cf} \kappa$ by hypothesis) the intersection of λ intervals of the form $(x, \infty]$ contains another of this same form. So we see that $\lambda\text{-}S = S$. In particular, $\Lambda(\lambda\text{-}S) \neq S$, so $\lambda\text{-}S$ does not have a continuous multiplication. (In this example, $\kappa\text{-}S$ is discrete.)

(ii) *Given any uncountable cardinal κ there is a semigroup S that satisfies the conditions of Lemma 1 and that has the property that every dense subset of S has cardinal at least κ, but for which $\aleph_0\text{-}S$ is jointly continuous.*

Let κ be an uncountable cardinal. We start with a semigroup T that has an identity 1 and satisfies (a) $\Lambda(T)$ contains a countable subset Z dense in T (and notice that we may take $1 \in Z$) and (b) every point of T has a countable neighborhood base. There do exist compact right topological semigroups with these properties that are not topological (for example, the semigroup S used in [1, Example 1], but with the first copy $\mathbb{T} = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ of the circle group replaced by $\{e^{in} : n \in \mathbb{Z}\} = \mathbb{Z}_0$ (say) to give $\mathbb{Z}_0 \cup \mathbb{T}_1 \cup \mathbb{T}_2$).

We consider the direct product T^κ with the direct product topology. Then $\Lambda(T^\kappa) = \Lambda(T)^\kappa \supseteq Z^\kappa$. However, we can find a smaller subset of $\Lambda(T^\kappa)$ that is dense in T^κ; this is the direct sum of κ copies of Z, a subset dense in the direct product, and it has cardinal exactly κ. Theorem 1 tells us that $\kappa\text{-}T^\kappa$ has continuous multiplication, but as in (i) the κ-topology is uninteresting since it is discrete. Moreover, no subset of T^κ with fewer than κ elements is dense.

We shall determine the $\aleph_0$-toplogy on T^κ. Let $(t_\alpha)_{\alpha<\kappa}$ be an element of T^κ. For each α, let $\{U_n(t_\alpha) : n = 1, 2 \ldots\}$ be a neighborhood base of t_α in T with $U_n(t_\alpha) \searrow \{t_\alpha\}$. For any finite subset F of κ with $\operatorname{card} F = r$, we write $V(F) = \prod_{\alpha<\kappa} V_\alpha$ where $V_\alpha = T$ for $\alpha \notin F$, $V_\alpha = U_r(t_\alpha)$ for $\alpha \in F$. If E is any countable set of predecessors of κ, we write

$$W(E) = \bigcap \{V(F) : F \text{ is a finite subset of } E\}.$$

Then $W(E)$ is an $\aleph_0$-neighborhood of (t_α). It is easy to see that in fact $W(E) = \prod_\alpha W_\alpha$ where $W_\alpha = T$ if $\alpha \notin E$, $W_\alpha = \{t_\alpha\}$ if $\alpha \in E$. It can now be seen that the $\aleph_0$-topology on T is determined by neighborhoods of the form $W(E)$.

It is not difficult to check directly that this topology makes multiplication in T^κ continuous. It is perhaps more illuminating to observe that if T_d is T with the discrete topology then $\aleph_0\text{-}(T_d)^\kappa$ is the same as $\aleph_0\text{-}T^\kappa$. Since multiplication in T_d^κ is continuous, so is multiplication in $\aleph_0\text{-}(T_d)^\kappa$ (the argument is as in the proof of Theorem 1). □

We conclude with a question. One of the difficult problems about compact right topological semigroups is to determine how the topological and algebraic structures interact. This is even true for minimal one-sided ideals though these have a simple algebraic structure. Thus, for a minimal left ideal L, the set $E(L)$ of idempotents in L is a left-zero semigroup ($ef = e$ for all e, f), the semigroups eL, for $e \in E(L)$, are isomorphic groups, and algebraically L is isomorphic to the direct product $E(L) \times (eL)$ [2, 1.3.11, 1.2.16]. Topologically L is compact. If S has a separately continuous multiplication, then L is isomorphic to the topological direct product of the compact subsemigroups $E(L)$ and (eL) [2, Theorem 1.5.1], but this may not be so in more general cases (see [8] for the semigroup $\beta\mathbb{N}$). Theorem 1 tells us that $\kappa\text{-}S$ is jointly continuous for some κ (though it is not compact); is it true that $\kappa\text{-}L$ is isomorphic to a topological direct product of $\kappa\text{-}E(L)$ and $\kappa\text{-}(eL)$?

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