PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

A STUDY ON A RELATION BETWEEN TWO SUMMABILITY METHODS

W. T. SULAIMAN

(Communicated by R. Daniel Mauldin)

ABSTRACT. Bor (1985, 1986) gave a relation between the two summability methods $|C, 1|_k$ and $|\overline{N}, p_n|_k$ of a series $\sum a_n$. These two methods are known to be independent. Generalizing the case, here we introduce relations between the two summability methods $|C, \alpha|_k$ and $|\overline{N}, p_n|_k$ using multipliers sequence $\{\varepsilon_n\}$.

1. INTRODUCTION

Let $\sum a_n$ be an infinite series of partial sums s_n. Let σ_n^δ and η_n^δ denote the nth Cesàro mean of order δ $(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\sum a_n$ is said to be absolutely summable (C, δ) with index k, or simply summable $|C, \delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty,$$

or equivalently,

$$\sum_{n=1}^{\infty} n^{-1}|\eta_n^\delta|^k < \infty.$$

Let $\{p_n\}$ be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty \ (P_{-1} = p_{-1} = 0).$$

A series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty \quad \text{(Bor [1])},$$

where

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v.$$

Received by the editors May 15, 1989 and, in revised forms, September 20, 1989, December 21, 1989, and March 27, 1990.
1991 *Mathematics Subject Classification*. Primary 40G99.

If we take $p_n = 1$, then $|\overline{N}, p_n|_k$ summability is equivalent to $|C, 1|_k$ summability. $|\overline{N}, p_n|_1$ is the same as $|\overline{N}, p_n|$. In general the two methods $|C, \delta|_k$ and $|\overline{N}, p_n|_k$ are not comparable.

Bor established the following two results:

Theorem A. *Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$*

$$\text{(I)} \qquad \begin{cases} \text{(i) } np_n = O(P_n), \\ \text{(ii) } P_n = O(np_n). \end{cases}$$

If $\sum a_n$ is summable $|C, 1|_k$, then it is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

Theorem B. *Let $\{p_n\}$ be a sequence of positive real constants such that it satisfies* (I). *If $\sum a_n$ is summable $|\overline{N}, p_n|_k$, then it is summable $|C, 1|_k$, $k \geq 1$.*

2. Main results

We prove the following:

Theorem 1. (A) *Let $\{p_n\}$ be a sequence of positive numbers. Let T_n be the $(\overline{N}, p_n)$-mean of the series $\sum a_n$. If*

$$\sum_{n=1}^{\infty} n^{k-1} |\varepsilon_n|^k |\Delta T_{n-1}|^k < \infty, \tag{2.1}$$

$$\sum_{n=1}^{\infty} n^{k-k\alpha-1} \left(\frac{P_n}{p_n}\right)^k |\varepsilon_n|^k |\Delta T_{n-1}|^k < \infty, \qquad (0 < \alpha < 1) \tag{2.2}$$

$$\sum_{n=1}^{\infty} n^{-1} \left(\frac{P_n}{p_n}\right)^k |\varepsilon_n|^k |\Delta T_{n-1}|^k < \infty, \qquad (\alpha \geq 1) \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{k-1} \left(\frac{P_n}{p_n}\right)^k |\Delta\varepsilon_n|^k |\Delta T_{n-1}|^k < \infty, \tag{2.4}$$

then the series $\sum a_n \varepsilon_n$ is summable $|C, \alpha|_k$, $k \geq 1$, $\alpha > 0$.

(B) *Let $\{p_n\}$ be a sequence of positive numbers such that* (I) *holds. Let $\{\lambda_n\}$, $\{\varepsilon_n\}$ be such that $\{\lambda_n\}$ is nonnegative, nondecreasing, $n^{1-\alpha}\lambda_n|\varepsilon_n| = O(1)$ for $0 < \alpha < 1$, $\lambda_n|\varepsilon_n| = O(1)$ and $\varepsilon_n = o(1)$ for $\alpha \geq 1$, $\Delta\varepsilon_n = O(n^{-1}|\varepsilon_n|)$, and*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta T_{n-1}|^k = O(\lambda_m^k), \qquad m \to \infty$$

then in order to have the series $\sum a_n \varepsilon_n$ summable $|C, \alpha|_k$, it is sufficient that

$$\sum_{n=1}^{\infty} n^{2-\alpha} \lambda_n |\Delta^2 \varepsilon_n| < \infty, \qquad (0 < \alpha < 1)$$

and

$$\sum_{n=1}^{\infty} n \lambda_n |\Delta^2 \varepsilon_n| < \infty, \qquad (\alpha \geq 1).$$

Theorem 2. (A) *Let* $\{p_n\}$ *be a sequence of positive numbers. Let* t_n^1 *be the nth* $(C, 1)$*-mean of the sequence* $\{na_n\}$. *If*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\varepsilon_n|^k |t_n^1|^k < \infty, \tag{2.5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left(\frac{P_n}{p_n}\right)^{k-1} |\varepsilon_n|^k |t_n^1|^k < \infty, \tag{2.6}$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta\varepsilon_n|^k |t_n^1|^k < \infty, \tag{2.7}$$

then the series $\sum a_n \varepsilon_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

(B) *Let* $\{p_n\}$ *be a sequence of positive numbers such that* (I) *holds. Let* $\{\lambda_n\}$, $\{\varepsilon_n\}$ *be such that* $\{\lambda_n\}$ *is nonnegative, nondecreasing,* $\lambda_n |\varepsilon_n| = O(1)$, $\varepsilon_n = o(1)$, $\Delta\varepsilon_n = O(n^{-1}|\varepsilon_n|)$, *and*

$$\sum_{n=1}^{m} n^{-1} |t_n^1|^k = O(\lambda_m^k),$$

then in order to have the series $\sum a_n \varepsilon_n$ *summable* $|\overline{N}, p_n|_k$, $k \geq 1$, *it is sufficient that*

$$\sum_{n=1}^{\infty} n \lambda_n |\Delta^2 \varepsilon_n| < \infty.$$

3. Required lemma

Lemma. *If* $\sigma > \delta > 0$, *then*

$$\sum_{n=v+1}^{m} \frac{(n-v)^{\delta-1}}{n^\sigma} = O(v^{\delta-\sigma}), \quad \text{as } m \to \infty.$$

4. Proof of the theorems

Proof of Theorem 1. (A) Let t_n^α be the nth (C, α)-mean, $\alpha > 0$, of the sequence $\{na_n \varepsilon_n\}$. Then we have

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \varepsilon_v,$$

where

$$A_n^\delta = \binom{n+\delta}{n} = \frac{(\delta+1)(\delta+2)\cdots(\delta+n)}{n!} \simeq \frac{n^\delta}{\Gamma(\delta+1)}, \qquad \delta \neq -1, -2, \dots.$$

As

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v,$$

then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v.$$

$$\begin{aligned}
t_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n} P_{v-1}a_v\{vA_{n-v}^{\alpha-1}P_{v-1}^{-1}\varepsilon_v\} \\
&= \frac{1}{A_n^\alpha}\left[\sum_{v=1}^{n-1}\left\{\sum_{r=1}^{v} P_{r-1}a_r\right\}\Delta\{vA_{n-v}^{\alpha-1}P_{v-1}^{-1}\varepsilon_v\} + \left\{\sum_{r=1}^{n} P_{r-1}a_r\right\} nP_{n-1}^{-1}\varepsilon_n\right] \\
&= -\frac{1}{A_n^\alpha}\left[\sum_{v=1}^{n-1}\left\{vA_{n-v}^{\alpha-1}\varepsilon_v\Delta T_{v-1} + \frac{P_{v-1}}{p_v}A_{n-v}^{\alpha-1}\varepsilon_v\Delta T_{v-1} + (v+1)\frac{P_{v-1}}{p_v}\right.\right. \\
&\qquad \times \Delta A_{n-v}^{\alpha-1}\varepsilon_v\Delta T_{v-1} \\
&\qquad \left.\left. + (v+1)\frac{P_{v-1}}{p_v}A_{n-v-1}^{\alpha-1}\Delta\varepsilon_v\Delta T_{v-1}\right\} + n\frac{P_n}{p_n}\varepsilon_n\Delta T_{n-1}\right] \\
&= t_{n,1}^\alpha + t_{n,2}^\alpha + t_{n,3}^\alpha + t_{n,4}^\alpha + t_{n,5}^\alpha .
\end{aligned}$$

In order to prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1}|t_{n,r}^\alpha|^k < \infty, \qquad r = 1, 2, 3, 4, 5.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-1}|t_{n,1}^\alpha|^k &\le \sum_{n=2}^{m+1}\frac{1}{nA_n^\alpha}\sum_{v=1}^{n-1} v^k A_{n-v}^{\alpha-1}|\varepsilon_v|^k|\Delta T_{v-1}|^k\left\{\sum_{v=1}^{n-1}\frac{A_{n-v}^{\alpha-1}}{A_n^\alpha}\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} v^k|\varepsilon_v|^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{A_{n-v}^{\alpha-1}}{nA_n^\alpha} \\
&= O(1)\sum_{v=1}^{m} v^k|\varepsilon_v|^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-1}}{n^{1+\alpha}} \\
&= O(1)\sum_{v=1}^{m} v^{k-1}|\varepsilon_v|^k|\Delta T_{v-1}|^k .
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-1}|t_{n,2}^\alpha|^k &\le \sum_{n=2}^{m+1}\frac{1}{nA_n^\alpha}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k A_{n-v}^{\alpha-1}|\varepsilon_v|^k|\Delta T_{v-1}|^k\left\{\sum_{v=1}^{n-1}\frac{A_{n-v}^{\alpha-1}}{A_n^\alpha}\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k|\varepsilon_v|^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{A_{n-v}^{\alpha-1}}{nA_n^\alpha} \\
&= O(1)\sum_{v=1}^{m} v^{-1}\left(\frac{P_v}{p_v}\right)^k|\varepsilon_v|^k|\Delta T_{v-1}|^k .
\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{m+1} n^{-1}|t_{n,4}^{\alpha}|^k &\leq \sum_{n=2}^{m+1} \frac{1}{nA_n^{\alpha}} \sum_{v=1}^{n-1} (v+1)^k \left(\frac{P_v}{p_v}\right)^k A_{n-v-1}^{\alpha-1} |\Delta\varepsilon_v|^k |\Delta T_{v-1}|^k \\ &\quad \times \left\{\sum_{v=1}^{n-1} \frac{A_{n-v-1}^{\alpha-1}}{A_n^{\alpha}}\right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^k \left(\frac{P_v}{p_v}\right)^k |\Delta\varepsilon_v|^k |\Delta T_{v-1}|^k \\ &\quad \times \sum_{n=v+1}^{m+1} \frac{A_{n-v-1}^{\alpha-1}}{nA_n^{\alpha}} \\ &= O(1) \sum_{v=1}^{m} v^{k-1} \left(\frac{P_v}{p_v}\right)^k |\Delta\varepsilon_v|^k |\Delta T_{v-1}|^k .\end{aligned}$$

$$\sum_{n=1}^{m} n^{-1}|t_{n,5}^{\alpha}|^k = O(1) \sum_{n=1}^{m} n^{k-k\alpha-1} \left(\frac{P_n}{p_n}\right)^k |\varepsilon_n|^k |\Delta T_{n-1}|^k .$$

For $t_{n,3}^{\alpha}$ the two cases $0 < \alpha < 1$, and $\alpha \geq 1$ need to be considered. For $0 < \alpha < 1$, we have

$$\begin{aligned}\sum_{n=2}^{m+1} n^{-1}|t_{n,3}^{\alpha}|^k &\leq \sum_{n=2}^{m+1} \frac{1}{n(A_n^{\alpha})^k} \sum_{v=1}^{n-1} (v+1)^k \left(\frac{P_v}{p_v}\right)^k |\Delta A_{n-v}^{\alpha-1}| |\varepsilon_v|^k |\Delta T_{v-1}|^k \\ &\quad \times \left\{\sum_{v=1}^{n-1} |\Delta A_{n-v}^{\alpha-1}|\right\}^{k-1} \\ &= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+k\alpha}} \sum_{v=1}^{n-1} v^k \left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-2} |\varepsilon_v|^k |\Delta T_{v-1}|^k \\ &\quad \times \left\{\sum_{v=1}^{n-1} (n-v)^{\alpha-2}\right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^k \left(\frac{P_v}{p_v}\right)^k |\varepsilon_v|^k |\Delta T_{v-1}|^k \\ &\quad \times \sum_{n=v+1}^{m+1} \frac{(n-v)^{\alpha-2}}{n^{1+k\alpha}} \\ &= O(1) \sum_{v=1}^{m} v^{k-k\alpha-1} \left(\frac{P_v}{p_v}\right)^k |\varepsilon_v|^k |\Delta T_{v-1}|^k .\end{aligned}$$

When $\alpha = 1$, $\Delta A_{n-v}^{\alpha-1} = 0$, hence $t_{n,3}^{\alpha} = 0$. It remains to consider $t_{n,3}^{\alpha}$ for

$\alpha > 1$, and for this case we have

$$\sum_{n=2}^{m+1} n^{-1}|t_{n,3}^{\alpha}|^k = O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+k\alpha}}\sum_{v=1}^{n-1} v^k\left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-2}\frac{|\varepsilon_v|^k|\Delta T_{v-1}|^k}{\cdot n^{(\alpha-1)(k-1)}}$$

$$\left(\text{as } \sum_{v=1}^{n-1}(n-v)^{\alpha-2} = O(1)\int_1^{n-1}(n-x)^{\alpha-2}\,dx = O(n^{\alpha-1}).\right)$$

$$= O(1)\sum_{v=1}^{m} v^k\left(\frac{P_v}{p_v}\right)^k |\varepsilon_v|^k|\Delta T_{v-1}|^k \times \sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-2}}{n^{k+\alpha}}$$

$$= O(1)\sum_{v=1}^{m} v^{-1}\left(\frac{P_v}{p_v}\right)^k |\varepsilon_v|^k|\Delta T_{v-1}|^k .$$

This completes the proof of (A). To prove (B), by (A), it is sufficient to show that the conditions (2.1)–(2.4) are all satisfied. Since $\Delta\varepsilon_n = O(n^{-1}|\varepsilon_n|)$, then by (I) each of the summations in (2.1) and (2.4) is equal to

$$O(1)\sum_{n=1}^{\infty} n^{-1}\left(\frac{P_n}{p_n}\right)^k |\varepsilon_n|^k|\Delta T_{n-1}|^k$$

$$= \begin{cases} O(1)\displaystyle\sum_{n=1}^{\infty} n^{k-k\alpha-1}\left(\frac{P_n}{p_n}\right)^k |\varepsilon_n|^k|\Delta T_{n-1}|^k & (0<\alpha<1), \\ O(1)\displaystyle\sum_{n=1}^{\infty} n^{-1}\left(\frac{P_n}{p_n}\right)^k |\varepsilon_n|^k|\Delta T_{n-1}|^k, & (\alpha \ge 1). \end{cases}$$

Therefore, only the two conditions (2.2) and (2.3) are left to be considered. For $0 < \alpha < 1$, we have

$$\sum_{v=1}^{m} v^{k-k\alpha-1}\left(\frac{P_v}{p_v}\right)^k |\varepsilon_v|^k|\Delta T_{v-1}|^k$$

$$= O(1)\sum_{v=1}^{m} v^{k-k\alpha}|\varepsilon_v|^k\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta T_{v-1}|^k$$

$$= O(1)\sum_{v=1}^{m-1}\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{k-1}|\Delta T_{r-1}|^k\{v^{k-k\alpha-1}|\varepsilon_v|^k + (v+1)^{k-k\alpha}\Delta|\varepsilon_v|^k\}$$

$$+ O(1)\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{k-1}|\Delta T_{r-1}|^k\cdot m^{k-k\alpha}|\varepsilon_m|^k$$

$$= I_1 + I_2 + I_3, \quad \text{say}.$$

$$
\begin{aligned}
I_1 &= O(1)\sum_{v=1}^{\infty} v^{k-k\alpha-1}\lambda_v^k|\varepsilon_v|^k = O(1)\sum_{v=1}^{\infty} v^{-\alpha}\lambda_v|\varepsilon_v| \\
&= O(1)\sum_{v=1}^{\infty} v^{-\alpha}\lambda_v\left|\sum_{r=v}^{\infty}\Delta\varepsilon_r\right|, \quad \text{as } \varepsilon_m = o(1) = O(1)\sum_{v=1}^{\infty} v^{-\alpha}\lambda_v\sum_{r=v}^{\infty}|\Delta\varepsilon_r| \\
&= O(1)\sum_{r=1}^{\infty}|\Delta\varepsilon_r|\sum_{v=1}^{r} v^{-\alpha}\lambda_v = O(1)\sum_{r=1}^{\infty}|\Delta\varepsilon_r|\lambda_r\int_1^r x^{-\alpha}\,dx \\
&= O(1)\sum_{r=1}^{\infty} r^{1-\alpha}\lambda_r|\Delta\varepsilon_r| = O(1)\sum_{r=1}^{\infty} r^{1-\alpha}\lambda_r\sum_{v=r}^{\infty}|\Delta^2\varepsilon_r|, \quad \text{as } \Delta\varepsilon_m = o(1) \\
&= O(1)\sum_{v=1}^{\infty}|\Delta^2\varepsilon_v|\sum_{r=1}^{v} r^{1-\alpha}\lambda_r \\
&= O(1)\sum_{v=1}^{\infty} v^{2-\alpha}\lambda_v|\Delta^2\varepsilon_v|.
\end{aligned}
$$

Since

$$\Delta|\varepsilon_v|^k = K(|\varepsilon_v| - |\varepsilon_{v+1}|)\xi^{k-1},$$

for some ξ between $|\varepsilon_v|$ and $|\varepsilon_{v+1}|$, by the mean value theorem,

$$
\begin{aligned}
\Delta|\varepsilon_v|^k &= O\{|\varepsilon_v|^{k-1}\Delta|\varepsilon_v|\}, \quad \text{as } \varepsilon_m = o(1) \\
&= O\{|\varepsilon_v|^{k-1}|\Delta\varepsilon_v|\},
\end{aligned}
$$

therefore,

$$
\begin{aligned}
I_2 &= O(1)\sum_{v=1}^{\infty} v^{k-k\alpha}\lambda_v^k\Delta|\varepsilon_v|^k \\
&= O(1)\sum_{v=1}^{\infty} v^{k-k\alpha}\lambda_v^k|\varepsilon_v|^{k-1}|\Delta\varepsilon_v| \\
&= O(1)\sum_{v=1}^{\infty} v^{1-\alpha}\lambda_v|\Delta\varepsilon_v| = O(1)\sum_{v=1}^{\infty} v^{1-\alpha}\lambda_v\sum_{r=v}^{\infty}|\Delta^2\varepsilon_r| \\
&= O(1)\sum_{r=1}^{\infty} r^{2-\alpha}\lambda_r|\Delta^2\varepsilon_r| \\
I_3 &= O(1)m^{k-k\alpha}\lambda_m^k|\varepsilon_m|^k = O(1).
\end{aligned}
$$

Now, for the case $\alpha \geq 1$ we have

$$
\begin{aligned}
&\sum_{v=1}^{m} v^{-1}\left(\frac{P_v}{p_v}\right)^k|\varepsilon_v|^k|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m}|\varepsilon_v|^k\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m-1}\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{k-1}|\Delta T_{r-1}|^k\Delta|\varepsilon_v|^k + O(1)\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{k-1}|\Delta T_{r-1}|^k|\varepsilon_m|^k \\
&= J_1 + J_2, \quad \text{say}.
\end{aligned}
$$

$$\begin{aligned} J_1 &= O(1)\sum_{v=1}^{\infty}\lambda_v^k|\varepsilon_v|^{k-1}|\Delta\varepsilon_v| = O(1)\sum_{v=1}^{\infty}\lambda_v|\Delta\varepsilon_v| \\ &= O(1)\sum_{v=1}^{\infty} v\lambda_v|\Delta^2\varepsilon_v|_v\lambda_v|\Delta^2\varepsilon_v|, \quad \text{as before}. \\ J_2 &= O(1)\lambda_m^k|\varepsilon_m|^k = O(1). \end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. (A) Let Q_n denote the $(\overline{N}, p_n)$-mean of the series $\sum a_n\varepsilon_n$. Then we have

$$Q_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\varepsilon_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\varepsilon_v$$

$$Q_n - Q_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\varepsilon_v .$$

$$\begin{aligned} Q_n - Q_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} va_v\frac{P_{v-1}\varepsilon_v}{v} \\ &= \frac{p_n}{P_nP_{n-1}}\left[\sum_{v=1}^{n-1}(v+1)t_v^1\left\{-\frac{p_v\varepsilon_v}{v} + \frac{P_v\varepsilon_v}{v(v+1)} + \frac{P_v\Delta\varepsilon_v}{v+1}\right\} \right. \\ &\qquad \left. + \frac{n+1}{n}P_{n-1}\varepsilon_n t_n^1\right] \\ &= Q_{n,1} + Q_{n,2} + Q_{n,3} + Q_{n,4}. \end{aligned}$$

In order to prove the theorem it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|Q_{n,r}|^k < \infty, \qquad r = 1, 2, 3, 4.$$

Applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|Q_{n,1}|^k &\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(1+\frac{1}{v}\right)^k p_v|\varepsilon_v|^k|t_v^1|^k\left\{\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right\}^{k-1} \\ &= O(1)\sum_{v=1}^{m} p_v|\varepsilon_v|^k|t_v^1|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\varepsilon_v|^k|t_v^1|^k . \end{aligned}$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|Q_{n,2}|^k \le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k p_v|\varepsilon_v|^k|t_v^1|^k\left\{\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k p_v|\varepsilon_v|^k|t_v^1|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^{k-1}|\varepsilon_v|^k|t_v^1|^k.$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|Q_{n,3}|^k \le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\varepsilon_v|^k|t_v^1|^k\left\{\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\varepsilon_v|^k|t_v^1|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\varepsilon_v|^k|t_v^1|^k.$$

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|Q_{n,4}|^k = O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\varepsilon_n|^k|t_n^1|^k.$$

The proof of (B) can be achieved exactly as in the case of Theorem 1. This completes the proof of Theorem 2.

5. Applications and corollaries

1. If we take $\alpha = 1$ and $\varepsilon_n = 1$ for all n in Theorem 1 (A), we get Theorem B provided (I) holds.
2. If we take $\alpha = 1$ and $\varepsilon_n = 1$ for all n in Theorem 2 (A), we get Theorem A provided (I) holds.

Corollary 1. *Let $\{p_n\}$ be a sequence of positive real constants such that $np_n = O(P_n)$. Then sufficient conditions that $\sum a_n\varepsilon_n$ be summable $|C, \alpha|_k$, $k \ge 1$, $\alpha > 0$, whenever $\sum a_n$ is summable $|\overline{N}, p_n|_k$ are*

(i) $|\varepsilon_n| = O\{n^{\alpha-1+1/k}(p_n/P_n)^{1/k}\}$ $(\alpha < 1)$,
(ii) $|\varepsilon_n| = O\{(np_n/P_n)^{1/k}\}$ $(\alpha \ge 1)$,
(iii) $|\Delta\varepsilon_n| = O\{n^{-1+1/k}(p_n/P_n)^{1/k}\}$.

Proof. Since $np_n = O(P_n)$, then, for $0 < \alpha < 1$,

$$\sum_{n=1}^{\infty} n^{k-1}|\varepsilon_n|^k|\Delta T_{n-1}|^k = O(1)\sum_{n=1}^{\infty} n^{k\alpha}\left(\frac{p_n}{P_n}\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta T_{n-1}|^k$$

$$= O(1)\sum_{n=1}^{\infty} n^{k\alpha-k}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta T_{n-1}|^k$$

$$= O(1)\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta T_{n-1}|^k = O(1).$$

Similarly, we can show that (2.2)–(2.4) are also satisfied, and the result follows by Theorem 1 (A).

Corollary 2. *Let* $\{p_n\}$ *be a sequence of positive real constants such that* $np_n = O(P_n)$. *Then sufficient conditions that* $\sum a_n \varepsilon_n$ *be summable* $|\overline{N}, p_n|_k$, $k \geq 1$, *whenever* $\sum a_n$ *is summable* $|C, 1|_k$ *are*

(i) $|\varepsilon_n| = O\{(np_n/P_n)^{1-1/k}\}$,
(ii) $|\Delta\varepsilon_n| = O\{n^{-1/k}(p_n/P_n)^{1-1/k}\}$.

Proof. Since $np_n = O(P_n)$, then $|\varepsilon_n| = O(1)$. Hence

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\varepsilon_n|^k |t_n^1|^k = O(1) \sum_{n=1}^{\infty} \frac{1}{n} |t_n^1|^k = O(1).$$

Conditions (2.6) and (2.7) are also satisfied, and the result follows by Theorem 2 (A).

Acknowledgment

I am so indebted to the referee who guided me for a good adjustment of this paper.

References

1. H. Bor, *On two summability methods*, Math. Proc. Cambridge Soc. **97** (1985), 147–149.
2. ——, *A note on two summability methods*, Proc. Amer. Math. Soc. **98** (1986), 81–84.

Department of Applied Sciences, College of Technological Studies, P.O. Box 42325, 70654 Kuwait