PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

EIGENVALUES OF SOME ALMOST PERIODIC FUNCTIONS

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(Communicated by Dennis Burke)

ABSTRACT. Let B_U be the set of real valued functions on R which are bounded and uniformly continuous. For $f, g \in B_U$, put

$$d(f, g) = \sup_{t \in R} |f(t) - g(t)|.$$

Then B_U becomes a metric space. On B_U we define a flow η by $\eta(f, t) = f_t$ for $(f, t) \in B_U \times R$. We denote the restriction of η to the hull of $f \in B_U$ by η_f. If f is almost periodic, then the set of eigenvalues of η_f coincides with the module of f (see J. Egawa, *Eigenvalues of compact minimal flows*, Math. Seminar Notes (Kobe Univ.), **10** (1982), 281–291. In this paper, we extend this result to almost periodic functions with some additional properties.

We denote the sets of real numbers and complex numbers by R and C, respectively. Let X be a metric space with metric d_X. A continuous mapping $T: X \times R \to X$ is called a *flow on* (*a phase space*) X, if T satisfies the following two conditions:

(1) $T(x, 0) = x$ for $x \in X$.
(2) $T(T(x, t), s) = T(x, t+s)$ for $x \in X$ and $t, s \in R$.

The orbit of T through $x \in X$ is denoted by $C_T(x)$, that is, $C_T(x) = \{T(x, t); t \in R\}$. The closure of a set $A \subset X$ is denoted by $\overline{A}$. A subset $M \subset X$ is called an *invariant set of* T if we have $C_T(x) \subset M$ for every $x \in M$. We denote the restriction of T to an invariant set M of T by $T|M$. A nonempty compact invariant set M of T is called a *minimal set of* T if we have $\overline{C_T(x)} = M$ for every $x \in M$. If X is itself a minimal set of T, we say that T is a *minimal flow on* X. We say that T is *equicontinuous*, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $x, y \in X$ with $d_X(x, y) < \delta$ and $t \in R$ we have $d_X(T(x, t), T(y, t)) < \varepsilon$.

Proposition 1. *Let T be a flow on a compact metric space X. If T is equicontinuous, then for every $x \in X$ $\overline{C_T(x)}$ is a minimal set of T.*

Proof. Easy.

Let T_n be flows on X_n $(n = 1, 2, \dots)$. We denote the product flow of $\{T_n\}$ on $\prod_{n=1}^{\infty} X_n$ by $\prod_{n=1}^{\infty} T_n$.

Received by the editors June 8, 1990 and, in revised form, December 3, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54H20.
Key words and phrases. Equicontinuous, minimal flow, almost periodic function, eigenvalues.

Proposition 2. *Let T_n be flows on compact metric spaces X_n $(n = 1, 2, \dots)$. If for every n T_n is equicontinuous, then the flow $\prod_{n=1}^{\infty} T_n$ is also equicontinuous.*

Proof. See [2, p. 27].

Let T be a minimal flow on a compact metric space X. $\alpha \in R$ is called an *eigenvalue of* T if there exists a continuous function $\chi_\alpha : X \to K$ such that $\chi_\alpha(T(x, t)) = \chi_\alpha(x)\exp(2\pi i\alpha t)$ for $(x, t) \in X \times R$, where K is the unit circle in the complex plane. In this case χ_α is called an *eigenfunction of T belonging to α*. We denote the set of eigenvalues of T by $\Lambda(T)$. We can see that $\Lambda(T)$ is an additive subgroup of R.

Proposition 3. *Let T be an equicontinuous minimal flow on a compact metric space X. Then for each pair of distinct points $x, y \in X$ there exists $\alpha \in \Lambda(T)$ such that $\chi_\alpha(x) \neq \chi_\alpha(y)$.*

Proof. By equicontinuity of T, we can define a group structure on X, and by this structure X becomes an Abelian topological group [7, p. 101]. We can easily see that every character of X with respect to this group structure is an eigenfunction of T. It follows that the proposition holds [6].

For the set $A \subset R$, we denote the least additive subgroup of R containing A by $\widetilde{A}$.

Proposition 4. *Let T be an equicontinuous minimal flow on a compact metric space X, and $A \subset \Lambda(T)$. If for each pair of distinct points $x, y \in X$ there exists $\alpha \in A$ such that $\chi_\alpha(x) \neq \chi_\alpha(y)$, then we have $\widetilde{A} = \Lambda(T)$.*

Proof. Since $\Lambda(T)$ is a subgroup of R, we have $\widetilde{A} \subset \Lambda(T)$. Let $C(X)$ be the set of complex valued continuous functions on X with the topology of uniform convergence, and $O \subset C(X)$ all of linear combinations of $\{\chi_\alpha\}_{\alpha \in \widetilde{A}}$. Then O is obviously a linear subspace of $C(X)$. Since for $\alpha, \beta \in \widetilde{A}$ we have $\chi_\alpha \cdot \chi_\beta = \chi_{\alpha+\beta}$, and we have $f \cdot g \in O$ for $f, g \in O$. Further, every constant function belongs to O. If $f \in O$, then $\overline{f} \in O$, because for every $\alpha \in \widetilde{A}$ we have $\overline{\chi}_\alpha = \chi_{-\alpha}$, where $\overline{f}$ is a complex conjugate of f. Hence by the assumption and Stone-Weierstrass' theorem [5, p. 119], O is dense in $C(X)$. Let μ be a unique invariant Borel measure of T (since T is equicontinuous, T is strictly ergodic [4, p. 510]), and $L^2(X, \mu)$ the set of square summable complex valued functions. We can easily see that

$$(\chi_\alpha, \chi_\beta) = \int_X \chi_\alpha(x)\overline{\chi_\beta(x)}\, d\mu(x) = 0$$

for $\alpha, \beta \in \Lambda(T)$ $(\alpha \neq \beta)$. Let $\beta \in \Lambda(T) - \widetilde{A}$. Then there exists a sequence $\{h_n\} \subset 0$ such that $h_n \to \chi_\beta$ uniformly as $n \to \infty$. Since $(\chi_\beta, h_n) = 0$ for every n by the above remark, we obtain $\lim_{n\to\infty}(\chi_\beta, h_n) = (\chi_\beta, \chi_\beta) = 0$. This means $\chi_\beta \equiv 0$. This is a contradiction. Hence we have $\widetilde{A} = \Lambda(T)$.

Proposition 5. *Let T_n be equicontinuous minimal flows on compact metric spaces X_n $(n = 1, 2, \dots)$. Put $X = \prod_{n=1}^{\infty} X_n$ and $T = \prod_{n=1}^{\infty} T_n$. Let $x \in X$, $M = \overline{C_T(x)}$, and $A = \bigcup_{n=1}^{\infty} \Lambda(T_n)$. Then we have $\widetilde{A} = \Lambda(T|M)$.*

Proof. By Propositions 1 and 2, M is a minimal set of T and $T|M$ is equicontinuous. Let $\alpha \in \Lambda(T_n)$. We denote the eigenfunction of T_n belonging to α by

$\chi_\alpha^{(n)}$. Define a function $\chi_\alpha\colon M \to K$ by $\chi_\alpha(x) = \chi_\alpha^{(n)}(x_n)$, where x_n is the nth coordinate of x. Then χ_α is continuous, and it is an eigenfunction of $T|M$ belonging to α. Hence $A \subset \Lambda(T|M)$. Let $x, y \in M$ $(x \neq y)$. Then there exists n such that $x_n \neq y_n$. By Proposition 3 there exists $\alpha \in \Lambda(T_n)$ such that $\chi_\alpha^{(n)}(x_n) \neq \chi_\alpha^{(n)}(y_n)$. Hence by Proposition 4 we have $\widetilde{A} = \Lambda(T|M)$.

We denote the n-dimensional Euclidean space by R^n. Let $B_U^{(n)}$ be the set of R^n-valued continuous functions on R which are bounded and uniformly continuous. For $f, g \in B_U^{(n)}$, put $d(f, g) = \sup_{t\in R} |f(t) - g(t)|$. Then $B_U^{(n)}$ becomes a metric space by this metric. On $B_U^{(n)}$ we define a flow η by $\eta(f, t) = f_t$ for $(f, t) \in B_U^{(n)} \times R$, where $f_t(s) = f(t+s)$ for $s \in R$. Then η is obviously equicontinuous. For $f \in B_U^{(n)}$, put $H(f) = \overline{C_\eta(f)} = \overline{\{f_t\}}_{t\in R}$, and we denote the restriction of η to $H(f)$ by η_f. A R^n-valued continuous function f on R is said to be *almost periodic* if for each $\varepsilon > 0$ there exists a relatively dense subset A_ε of R such that $|f(t+\tau) - f(t)| < \varepsilon$ for $t \in R$ and $\tau \in A_\varepsilon$. We denote the set of R^n-valued almost periodic functions on R by $AP^{(n)}$. Then the following proposition is known [3].

Proposition 6. *Let $f \in AP^{(n)}$. Then*

(1) $f \in B_U^{(n)}$.
(2) *$H(f)$ is compact, and hence η_f is an equicontinuous minimal flow on $H(f)$.*
(3) *For each $\alpha \in R$, $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\exp(-2\pi i\alpha s)\,ds$ exists.*

For $f \in AP^{(n)}$, put $\Lambda_f = \{\alpha \in R;\ \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\exp(-2\pi i\alpha s)\,ds \neq 0\}$.

Proposition 7. *Let $f \in AP^{(n)}$. Then $\widetilde{\Lambda}_f = \Lambda(\eta_f)$.*

Proof. We sketch the proof for $n = 1$ (see [1] for detail). Define a function $F\colon H(f) \to R$ by $F(g) = g(0)$ for $g \in H(f)$. Then F is continuous on $H(f)$ and $F(\eta_f(g, t)) = F(g_t) = g_t(0) = g(t)$ for $(g, t) \in H(f) \times R$. Let Δ be a set of eigenfunctions of η_f. Then Δ coincides with the set of characters of the associated topological group $H(f)$. Let $\lambda \notin \Lambda(\eta_f)$, and $\varepsilon > 0$. Then there exist $\{a_j\}_{j=1}^N \subset C$ and $\{\chi_{\lambda_j}\} \subset \Delta$ such that

$$\left| F(g) - \sum_{j=1}^N a_j \chi_{\lambda_j}(g) \right| < \varepsilon$$

for $g \in H(f)$ [6]. Since

$$\frac{1}{t}\int_0^t \left| F(f_s) - \sum_{j=1}^N a_j \chi_{\lambda_j}(f_s) \right| ds < \varepsilon$$

for every $t \in R$, we have

$$\varepsilon \geq \left| \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\exp(-2\pi i\lambda s)\,ds \right|,$$

because $\lambda \neq \lambda_j$ $(j = 1, 2, \dots, N)$ and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \exp(-2\pi i(\lambda - \lambda_j))\,ds = 0.$$

Hence, since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(s) \exp(-2\pi i\lambda s)\, ds = 0.$$

This implies $\lambda \notin \Lambda_f$, which implies $\Lambda_f \subset \Lambda(\eta_f)$. Further, we shall show Λ_f satisfies the condition of Proposition 4. Let $\Lambda_f = \{\lambda_{i_j}\} \subset \Lambda(\eta_f)$ and χ_{i_j} be an eigenfunction of η_f belonging to λ_{i_j}. Then there exists a sequence $\{f_n\}$ such that

$$f_n(t) = \sum_{j=1}^{l_n} a_j^{(n)} \exp(2\pi i\lambda_{i_j} t) \qquad (a_j^{(n)} \in C)$$

converges to f uniformly on R as $n \to \infty$ [3, p. 48]. Put

$$F_n(g) = \sum_{j=1}^{l_n} a_j^{(n)} \chi_{i_j}^{-1}(f) \chi_{i_j}(g)$$

for $g \in H(f)$. Then F_n is continuous on $H(f)$, and

$$|F_n(f_t) - F(f_t)| = |f^{(n)}(t) - f(t)|$$

for $t \in R$. Since $\{f_t\}_{t\in R}$ is dense in $H(f)$, $\{F_n\}$ converges to F uniformly on $H(f)$. We assume that $\chi_{i_j}(g) = \chi_{i_j}(h)$ $(g, h \in H(f))$ for all j. Then we have $g(t) = F(g_t) = \lim_{n\to\infty} F_n(g_t) = \lim_{n\to\infty} F_n(h_t) = F(h_t) = h(t)$ for $t \in R$. Hence $g = h$, which implies that Λ_f satisfies the condition of Proposition 4. Hence we have $\widetilde{\Lambda}_f = \Lambda(\eta_f)$.

For $n \geq 2$ we can easily prove the proposition for considering product flows.

Proposition 8. *Let $A_n \subset R$ $(n = 1, 2, \dots)$, $A = \bigcup_{n=1}^{\infty} A_n$, and $B = \bigcup_{n=1}^{\infty} \widetilde{A}_n$. Then we have $\widetilde{A} = \widetilde{B}$.*

Proof. Easy.

Let T and S be flows on X and Y, respectively. A continuous mapping $h\colon X \to Y$ is called a *homomorphism from T to S* if $H(T(x, t)) = S(h(x), t)$ holds for $(x, t) \in X \times R$. Further, if h is a homeomorphism from X onto Y, then we say that h is an *isomorphism from T to S*.

Proposition 9. *Let T and S be minimal flows on compact metric spaces X and Y, respectively. If there exists an isomorphism from T to S, then we have $\Lambda(T) = \Lambda(S)$.*

Proof. Easy.

Let $W \subset R^n$ be an open set, and $BU^{(n)}(W)$ the set of continuous functions from $W \times R$ to R^n which satisfy the following condition: $f \in BU^{(n)}(W)$ if and only if for each compact set $K \subset W$ f is bounded and uniformly continuous on $K \times R$. We define a metric on $BU^{(n)}(W)$ in the following way. Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of compact subsets of W such that $K_m \subset K_{m+1}$ $(m = 1, 2, \dots)$ and $W = \bigcup_{m=1}^{\infty} K_m$. For $f, g \in BU^{(n)}(W)$, put

$$d_m(f, g) = \sup_{(x,t)\in K_m \times R} \{|f(x, t) - g(x, t)|\},$$

$$\rho_m(f, g) = \frac{d_m(f, g)}{1 + d_m(f, g)},$$

and

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(f, g).$$

Then $BU^{(n)}(W)$ becomes a metric space by this ρ. On $BU^{(n)}(W)$ we define a flow ξ by $\xi(f, t) = f_t$ for $(f, t) \in BU^{(n)}(W) \times R$, where $f_t(x, s) = f(x, t+s)$ for $(x, s) \in W \times R$. It is easy to verify that it is well defined. Obviously ξ is an equicontinuous flow on $BU^{(n)}(W)$. For $f \in BU^{(n)}(W)$, put $\Omega(f) = \overline{C_\xi(f)} = \overline{\{f_t\}_{t \in R}}$ and $\xi_f = \xi | \Omega(f)$. A continuous function $f\colon W \times R \to R^n$ is said to be *almost periodic in t uniformly for $x \in W$* if for each compact set $K \subset W$ and $\varepsilon > 0$ there exists a relatively dense subset $A_{K\varepsilon}$ of R such that $\tau \in A_{K\varepsilon}$ and $(x, t) \in K \times R$ imply $|f(x, t+\tau) - f(x, t)| < \varepsilon$. The set of continuous functions which are almost periodic in t uniformly for $x \in W$ is denoted by $AP^{(n)}(W)$. The following proposition is known [3].

Proposition 10. *Let $f \in AP^{(n)}(W)$. Then*

(1) $f \in BU^{(n)}(W)$.

(2) *$\Omega(f)$ is compact, and hence ξ_f is an equicontinuous minimal flow on $\Omega(f)$.*

(3) *for $x \in W$ and $\alpha \in R$,*

$$\lambda(\alpha, x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(x, s) \exp(-2\pi i \alpha s)\, ds$$

exists and $\lambda(\alpha, \cdot)$ is continuous on W.

For $f \in AP^{(n)}(W)$, put $\Lambda_f(W) = \{\alpha \in R; \lambda(\alpha, x) \not\equiv 0\}$. Then we have the following theorem, which is the main theorem in this paper.

Theorem. *Let $f \in AP^{(n)}(W)$. Then we have $\widetilde{\Lambda}_f(W) = \Lambda(\xi_f)$.*

Proof. Let $a \in W$, and $k_a(t) = f(a, t)$ for $t \in R$. Then $k_a \in AP^{(n)}$ and $H(k_a) = \{g(a, \cdot); g \in \Omega(f)\}$. Put $H(k_a) = H_a(f)$. Define a mapping h_a from $\Omega(f)$ to $H_a(f)$ by $h_a(g) = g(a, \cdot)$ for $g \in \Omega(f)$. Then h_a is obviously continuous and a homomorphism from ξ_f to $\eta | H_a(f)$. Let $\{a_m\}_{m=1}^{\infty}$ be a dense subset of W. Put $Y = \prod_{n=1}^{\infty} H_{a_m}(f)$ and $T = \prod_{m=1}^{\infty} \eta | H_{a_m}(f)$. Then T is an equicontinuous flow on the compact metric space Y by Proposition 2, because $\eta | H_{a_m}(f)$ is equicontinuous for every n. Define a mapping h from $\Omega(f)$ to Y by $h(g) = (h_{a_m}(g))$ for $g \in \Omega(f)$. Then h is continuous and a homomorphism from ξ_f to T. Further, we can easily see that h is injection. Put $M = h(\Omega(f))$. Then we have $M = \overline{C_T(h(f))}$. Hence we have $\Lambda(T|M) = \widetilde{A}$ by Proposition 5, where $A = \bigcup_{m=1}^{\infty} \Lambda(\eta | H_{a_m}(f))$. Put $A_m = \{\alpha \in R; \lim_{t \to \infty} \frac{1}{t} \int_0^t f(a_m, s) \exp(-2\pi i \alpha s)\, ds \neq 0\}$. Then $\Lambda_f(W) = \bigcup_{m=1}^{\infty} A_m$. In fact, $\bigcup_{m=1}^{\infty} A_m \subset \Lambda_f(W)$ is obvious. Let $\alpha \notin \bigcup_{m=1}^{\infty} A_m$. Then $\lambda(\alpha, a_m) = 0$ for each m. Since $\lambda(\alpha, \cdot)$ is continuous on W and $\{a_m\}_{m=1}^{\infty}$ is dense in W, we have $\lambda(\alpha, x) \equiv 0$. This means that $\alpha \notin \Lambda_f(W)$, which implies $\Lambda_f(W) = \bigcup_{m=1}^{\infty} A_m$. Since $\Lambda(\eta | H_{a_m}(f)) = \widetilde{A}_m$ by Proposition 7, we have $\Lambda(T|M) = \widetilde{\Lambda}_f(W)$ by Proposition 8. Since ξ_f is isomorphic to $T|M$, we obtain $\Lambda(\xi_f) = \Lambda(T|M) = \widetilde{\Lambda}_f(W)$ by Proposition 9.

Corollary. *Let* $f, g \in AP^{(n)}(W)$. *Then* ξ_f *and* ξ_g *are isomorphic if and only if* $\widetilde{\Lambda}_f(W) = \widetilde{\Lambda}_g(W)$.

Proof. Since $\Lambda(\xi_f) = \widetilde{\Lambda}_f(W) = \widetilde{\Lambda}_g(W) = \Lambda(\xi_g)$ by the theorem and ξ_f and ξ_g are equicontinuous, the corollary follows.

References

1. J. Egawa, *Eigenvalues of compact minimal flows*, Math. Seminar Notes (Kobe Univ.) **10** (1982), 281–291.
2. R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.
3. A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Math., vol. 377, Springer-Verlag, New York, 1974.
4. V. V. Nemytskii and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Univ. Press, Princeton, NJ, 1960.
5. M. Nibu, *Topological groups*, Iwanami, Tokyo, 1976. (Japanese)
6. L. S. Pontryagin, *Topological groups*, (Japanese Transl.) Iwanami, Tokyo, 1957.
7. G. R. Sell, *Topological dynamics and ordinary differential equations*, Van Nostrand Math. Studies, no. 33, Von Nostrand Reinhold, London, 1971.

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