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A GENERALIZATION OF D'ALEMBERT'S FUNCTIONAL EQUATION

STANLEY GUDDER

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ABSTRACT. Recent investigations in the foundations of physics have shown that a quantum mechanical influence function satisfies a generalized d'Alembert functional equation. The present paper derives the solutions for this equation on a topological linear space.

1. INTRODUCTION

D'Alembert's equation is one of the most important functional equations [1–4, 7]. This equation was originally introduced as a tool for studying the vibrating string problem and for axiomatic investigations of the parallelogram law for addition of force vectors [5, 6]. It has been applied in noneuclidean mechanics [4] and harmonic analysis [10]. A function $u\colon \mathbb{R} \to \mathbb{R}$ is said to satisfy d'Alembert's equation if

$$u(x+y) + u(x-y) = 2u(x)u(y) \tag{1.1}$$

for all x, $y \in \mathbb{R}$. A classical result states that the only continuous solutions of (1.1) are $u(x) \equiv 0$, $u(x) = \cos ax$, and $u(x) = \cosh ax$ [2, 4].

Recent investigations in the mathematical foundations of quantum mechanics have resulted in a generalization of (1.1). In his studies of discrete models for classical and quantum physics, Hemion introduced the concept of an influence $u\colon \mathbb{R} \to \mathbb{R}$ [8, 9]. The function u provides a measure of the influence between physical states (or configurations). By applying the principle of strong causality (the future cannot influence the past), Hemion has argued that u must satisfy the following causal condition

$$\sum_{i=1}^{n} u(y_i) = 0 \Rightarrow \sum_{i=1}^{n} [u(x+y_i) + u(x-y_i)] = 0 \tag{1.2}$$

for all $x \in \mathbb{R}$. It is clear that (1.1) implies (1.2), so (1.2) provides an interesting generalization of d'Alembert's equation. A physical interpretation of (1.2) states that if a present total influence vanishes, then it still vanishes when future effects are included.

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We shall find it useful to introduce the following closely related condition:

$$\begin{aligned}\sum_{i=1}^{n} u(x_i) = \sum_{i=1}^{m} u(y_i) &\Rightarrow \sum_{i=1}^{n}[u(x+x_i)+u(x-x_i)] \\ &= \sum_{i=1}^{m}[u(x+y_i)+u(x-y_i)]\end{aligned} \tag{1.3}$$

for all $x \in \mathbb{R}$. This condition can be given a physical interpretation similar to that of (1.2). Of course (1.3) is also a generalization of (1.1). It will follow from our work that (1.2) and (1.3) are equivalent if u is continuous and has a zero. If in addition, $u(0) = 1$, then (1.1)–(1.3) are all equivalent.

Hemion has shown that if $u\colon \mathbb{R} \to \mathbb{R}$ is periodic with period 2π, even, continuous, monotone on $[0, \pi]$, satisfies a condition slightly stronger than (1.2), and $u(0) = -u(\pi) = 1$, then $u(x) = \cos x$ for all $x \in \mathbb{R}$. In this paper, we present a vast generalization of Hemion's result. We cannot only dispense with many of Hemion's conditions, but our theorem applies to a real topological linear space V and gives a classification of solutions that is similar to the classical result. Although the main application of our theorem is for the case $V = \mathbb{R}$, it may also be important when V is a Hilbert space (Corollary 4) since quantum states are frequently represented by unit vectors in a Hilbert space.

2. Solutions of the functional equation

We now present our main result together with some corollaries. Let V be a real topological linear space. A subset $A \subseteq V$ is *balanced* if $tA \subseteq A$ for $|t| \leq 1$ and *absorbing* if for every $x \in V$ there exists an $\varepsilon > 0$ such that $tx \in A$ for $|t| < \varepsilon$. Recall that every neighborhood of 0 is absorbing and includes a balanced neighborhood of 0 [13]. We say that a map $u\colon V \to \mathbb{R}$ is C1 (C2) if u satisfies (1.2) [(1.3)] for every $x \in V$.

Theorem 1. *If $u\colon V \to \mathbb{R}$ is continuous and* C2*, then there exists a continuous linear functional $f\colon V \to \mathbb{R}$ such that either $u(x) = u(0)\cos f(x)$ for all $x \in V$ or $u(x) = u(0)\cosh f(x)$ for all $x \in V$. Moreover, f is unique in the sense that if $u \neq 0$ and $u(x) = u(0)\cos g(x)$ or $u(x) = u(0)\cosh g(x)$ for a continuous linear functional g, then $g = \pm f$.*

Proof. We may assume $u \neq 0$, since otherwise, we are finished. We may also assume $u \neq 1$, since otherwise, we let $f = 0$ and again we are finished. If $u(0) = 0$, then $u(0) + u(0) = u(0)$. Applying (1.3) gives $4u(x) = 2u(x)$ for every $x \in V$. Hence, $u = 0$, which is a contradiction, so $u(0) \neq 0$. We may assume that $u(0) = 1$, since otherwise, we could consider $v = u/u(0)$ and v satisfies the hypotheses of the theorem. We now consider two cases, either u has a zero or does not.

Suppose u has a zero so that there exists a $z \in V$ such that $u(z) = 0$. Since $u(z) + u(z) = u(z)$, applying (1.3) gives

$$u(x+z) + u(x-z) = 0 \tag{2.1}$$

for all $x \in V$. We now show that u is C1. If $\sum u(y_i) = 0$, then $\sum u(y_i) = u(z)$. Applying (1.3) and (2.1) gives

$$\sum[u(x+y_i)+u(x-y_i)] = u(x+z)+u(x-z) = 0$$

for every $x \in V$. Hence, (1.2) holds so u is C1. Letting $x = z$ in (2.1) gives $u(2z) = -1$, and replacing x by $x - z$ in (2.1) gives $u(x - 2z) = -u(x)$ for all $x \in V$. Assume that $u(y)$ is rational and $u(y) = s/t$ for integers s, t with $t \neq 0$. Suppose $u(y) > 0$, in which case we can assume that $s > 0$, $t > 0$. Let $n = s + t$, $y_1 = \cdots = y_s = 2z$, $y_{s+1} = \cdots = y_n = y$. Then

$$\sum_{i=1}^{n} u(y_i) = \sum_{i=1}^{s} u(y_i) + \sum_{i=s+1}^{n} u(y_i) = -s + tu(y) = 0.$$

Applying (1.2) we have for every $x \in V$

$$\begin{aligned} 0 &= \sum_{i=1}^{n} [u(x + y_i) + u(x - y_i)] \\ &= s[u(x + 2z) + u(x - 2z)] + t[u(x + y) + u(x - y)] \\ &= -2su(x) + t[u(x + y) + u(x - y)]. \end{aligned}$$

Hence,

$$u(x + y) + u(x - y) = (2s/t)u(x) = 2u(x)u(y). \tag{2.2}$$

Of course, this is d'Alembert's equation. If $u(y) = 0$ then (2.2) follows directly from (1.2). Now suppose $u(y) < 0$, in which case we can assume that $t > 0$ and $s < 0$. Let $n = |s| + t$, $y_1 = \cdots = y_{|s|} = 0$, $y_{|s|+1} = \cdots = y_n = y$. Then $\sum_{i=1}^{n} u(y_i) = |s| + tu(y) = 0$. Applying (1.2) gives for every $x \in V$

$$0 = \sum_{i=1}^{n} [u(x + y_i) + u(x - y_i)] = 2|s|u(x) + t[u(x + y) + u(x - y)].$$

Hence,

$$u(x + y) + u(x - y) = -(2|s|/t)u(x) = 2u(x)u(y).$$

We conclude that (2.2) holds whenever $u(y)$ is rational. Letting $x = y$ in (2.2) gives

$$1 + u(2y) = 2[u(y)]^2 \tag{2.3}$$

whenever $u(y)$ is rational.

We now show that u is not identically 1 on any neighborhood N of 0. If it were then applying (2.3) for $y \in N$ gives $u(2y) = 2[u(y)]^2 - 1 = 1$, so u is identically 1 on $2N$. Continuing, u is identically 1 on $2^n N$ for every positive integer n. Since N is absorbing, we conclude that $u = 1$ on V, which is a contradiction. We now show that for any neighborhood N of 0 there exists a $y \in N$ such that $u(y)$ is rational and $u(y) \neq 1$. Indeed, let $M \subseteq N$ be a balanced neighborhood of 0. Since u is not identically 1 on M, there exists an $x \in M$ such that $u(x) \neq 1$. Now the set $A = \{tx : 0 \leq t \leq 1\}$ is connected and $u|A$ is continuous. Hence, $u(A)$ is connected so u attains every value between 1 and $u(x)$ (the intermediate value theorem). Hence, there is a $y \in A$ such that $u(y)$ is rational and $u(y) \neq 1$. We next show that u cannot be identically a constant $c \neq 0$ on any neighborhood. Indeed, suppose $u(x) = c \neq 0$ for all $x \in N(w)$ when $N(w)$ is a neighborhood of w. Now there exists a balanced neighborhood N of 0 such that $N + w \subseteq N(w)$, and there is a $y \in N$ such that $u(y)$ is rational and $u(y) \neq 1$. Then $w + y$, $w - y \in N(w)$, so by (2.2)

$$2c = u(w+y) + u(w-y) = 2u(w)u(y) = 2cu(y).$$

Hence $c = 0$, which is a contradiction.

Now suppose $u(y)$ is irrational and $N(y)$ is a neighborhood of y. Since u is nonconstant on $N(y)$, as before, by the intermediate value theorem, there is a $w \in N(y)$ such that $u(w)$ is rational. In this way we obtain a net w_α such that $w_\alpha \to y$ and $u(w_\alpha)$ are rational. Since u is continuous, by (2.2) we have for every $x \in V$

$$\begin{aligned} u(x+y) + u(x-y) &= \lim[u(x+w_\alpha) + u(x-w_\alpha)] \\ &= \lim 2u(x)u(w_\alpha) = 2u(x)u(y). \end{aligned}$$

We conclude that (2.2) holds for every x, $y \in V$ and, in particular, (2.3) holds for every $y \in V$. Letting $x = 0$ in (2.2) gives $u(-y) = u(y)$ for all $y \in V$, so u is even.

Define $g\colon V \to \mathbb{C}$ by $g(x) = u(x) + iu(x+z)$ where z is a zero of u. Then for every x, $y \in V$, applying (2.2), (2.1) and evenness gives

$$\begin{aligned} g(x)g(y) &= u(x)u(y) - u(x+z)u(y+z) + i[u(y)u(x+z) + u(x)u(y+z)] \\ &= u(x+y) + iu(x+y+z) = g(x+y). \end{aligned}$$

Moreover, by (2.3) we have

$$|g(x)|^2 = [u(x)]^2 + [u(x+z)]^2 = 1.$$

Hence, g is a continuous character on V. It follows from a result in [11] that there exists a continuous linear functional $f\colon V \to \mathbb{R}$ such that $g(x) = e^{if(x)}$ for all $x \in V$. Hence $u(x) = \operatorname{Re} g(x) = \cos f(x)$. In the general case, when $u(0) \neq 1$ we obtain $u(x) = u(0)\cos f(x)$.

For the second case, u has no zero. Suppose there is a $w \in V$ such that $u(w) < 0$. Now there exists a balance neighborhood N of 0 such that $w \in N$. Since $u(0) = 1$, by the intermediate value theorem there is a $y \in N$ such that $u(y) = 0$. This gives a contradiction, so $u(w) > 0$ for all $w \in V$. Now assume that $u(y) = s/t$ is rational, where s, t are positive integers. Let $x_1 = \cdots = x_s = 0$, $y_1 = \cdots = y_t = y$. Then

$$\sum_{i=1}^{t} u(y_i) = tu(y) = su(0) = \sum_{i=1}^{s} u(x_i).$$

Since u is C2, applying (1.3) gives for every $x \in V$, $t[u(x+y) + u(x-y)] = 2su(x)$. Hence (2.2) again holds whenever $u(y)$ is rational. By the same argument used before, we conclude that (2.2) and (2.3) hold for every x, $y \in V$.

We now show that $u(x) \geq 1$ for all $x \in V$. Let $a_1 = 2^{-1/2}$, and for any integer $n > 1$ let $a_n = [(a_{n-1} + 1)/2]^{1/2}$. We prove by induction on n that $u(x) \geq a_n$ for all $x \in V$. Applying (2.3) we have

$$[u(x)]^2 + \tfrac{1}{2} + \tfrac{1}{2}u(2x) \geq \tfrac{1}{2}.$$

Hence $u(x) \geq a_1$ for all $x \in V$, so the result holds for $n = 1$. Suppose the result holds for the integer $n \geq 1$. Applying (2.3) gives

$$[u(x)]^2 = \tfrac{1}{2} + \tfrac{1}{2}u(2x) \geq \tfrac{1}{2}(1 + a_n),$$

so $u(x) \geq a_{n+1}$ for all $x \in V$, which completes the induction proof. We next show that $a_n \leq 1$ for all n. Indeed, $a_1 \leq 1$ and if $a_n \leq 1$ then $a_{n+1}^2 = (a_n + 1)2 \leq 1$, so the result follows by induction. Finally, a_n is an increasing sequence since

$$a_n = [\tfrac{1}{2}(a_{n-1} + 1)]^{1/2} \geq [\tfrac{1}{2}(a_{n-1} + a_{n-1})]^{1/2} = (a_{n-1})^{1/2} \geq a_{n-1}.$$

Letting $L = \lim a_n$, it follows that $L^2 = (L+1)/2$. Solving this equation gives $L = 1$. We conclude that $u(x) \geq L = 1$ for all $x \in V$.

Since $u \neq 1$, there exists a $w \in V$ such that $u(w) > 1$. Let $c = [2(u(w) - 1)]^{-1/2}$ and define $g\colon V \to \mathbb{R}$ by

$$g(x) = u(x) + c[u(x + w/2) - u(x - w/2)]. \tag{2.4}$$

Then a straightforward calculation (see [4, p. 220]) gives $g(x+y) = g(x)g(y)$ for all x, $y \in V$. It follows that $g(2x) = [g(x)]^2 \geq 0$ so $g(x) \geq 0$ for all $x \in V$. In fact, $g(x) > 0$ for all $x \in V$. Indeed, if $g(y) = 0$ then $g(x+y) = 0$ for all $x \in V$. In particular, $g(0) = 0$ but by (2.4), $g(0) = 1$, which is a contradiction. Letting $f(x) = \log g(x)$, we have $f(x+y) = f(x)+f(y)$ for all x, $y \in V$. Since f is continuous, we conclude that f is a continuous linear functional on V. From (2.4) we have

$$g(-x) = u(x) - c[u(x + w/2) - u(x - w/2)].$$

Hence

$$u(x) = \frac{g(x) + g(-x)}{2} = \frac{e^{f(x)} + e^{-f(x)}}{2} = \cosh f(x).$$

Again, in the general case, we have $u(x) = u(0)\cosh f(x)$.

For uniqueness, suppose $u \neq 0$. If $u = 1$ then $f = 0$ is the unique solution, so suppose $u \neq 1$. Clearly, if $g\colon V \to \mathbb{R}$ is a continuous linear functional, we cannot have

$$u(x) = u(0)\cos f(x) = u(0)\cosh g(x),$$

so suppose

$$u(x) = u(0)\cos f(x) = u(0)\cos g(x) \tag{2.5}$$

for all $x \in V$. Now there exists a neighborhood N of 0 such that $|f(x)|$, $|g(x)| < \pi/2$ for all $x \in N + N$ [13]. It follows from (2.5) that if $w \in N + N$ then $g(w) = \pm f(w)$. Now suppose x, $y \in N$ and $g(x) = f(x) \neq 0$ while $g(y) = -f(y) \neq 0$. Then

$$g(x + y) = g(x) + g(y) = f(x) - f(y).$$

Now suppose $g(x+y) = f(x+y)$. Then $f(x)+f(y) = f(x)-f(y)$, which gives the contradiction $f(y) = 0$. Similarly, if $g(x+y) = -f(x+y)$, we obtain the contradiction $f(x) = 0$. Hence either $g(x) = f(x)$ for $g(x) = -f(x)$ for all $x \in N$. Suppose the first case holds and $w \in V$. Since N is absorbing, there is a $t \in \mathbb{R} \setminus \{0\}$ such that $tw \in N$. Hence $tg(w) = g(tw) = f(tw) = tf(w)$, so $g = f$. Similarly, in the second case, $g = -f$. The proof is similar for the case

$$u(x) = u(0)\cosh f(x) = u(0)\cosh g(x). \quad \square$$

Corollary 2. *If $u\colon V \to \mathbb{R}$ is continuous,* C1, *and has a zero, then there exists a continuous linear functional $f\colon V \to \mathbb{R}$ such that $u(x) = u(0)\cos f(x)$.*

Proof. This follows from the first half of the proof of Theorem 1. □

Corollary 3. *Let V be a real Hilbert space with inner product $\langle\cdot\,,\cdot\rangle$. If $u\colon V \to \mathbb{R}$ is continuous and* C2, *then there exists a $y \in V$ such that $u(x) = u(0)\cos\langle x\,, y\rangle$ or $u(x) = u(0)\cosh\langle x\,, y\rangle$ for all $x \in V$. Moreover, y is unique in the sense that if $u \neq 0$ and either $u(x) = u(0)\cos\langle x\,, z\rangle$ or $u(x) = u(0)\cosh\langle x\,, z\rangle$ for all $x \in V$, then $z = \pm y$.*

Proof. This follows from Theorem 1 since every continuous linear functional on V has the form $f(x) = \langle x\,, y\rangle$ for some $y \in V$. □

Corollary 4. *Let V be a real Hilbert space with inner product $\langle\cdot\,,\cdot\rangle$. If $u\colon V \to \mathbb{R}$ is continuous,* C1, *and has a zero, then there exists a $y \in V$ such that $u(x) = u(0)\cos\langle x\,, y\rangle$. If $u \neq 0$ then the elements $\pm\pi y/2\|y\|^2$ are the unique zeros of u with smallest norm.*

Proof. We only need to prove the last statement. Notice that $y \neq 0$ since u has a zero. Letting $z = \pi y/2\|y\|^2$, we have

$$u(\pm z) = u(0)\cos(\pm\pi/2) = 0\,,$$

so $\pm z$ are zeros of u. Let w be an arbitrary zero of u. Then $0 = u(w) = u(0)\cos\langle w\,, y\rangle$. It follows that $\langle w\,, y\rangle = n\pi/2$ where n is an odd integer. By Schwarz's inequality

$$|n|\pi/2 = |\langle w\,, y\rangle| \leq \|w\|\,\|y\|\,.$$

Hence

$$\|w\| \geq |n|\pi/2\|y\| = |n|\,\|z\| \geq \|z\|\,, \tag{2.6}$$

so $\pm z$ are zeros with minimal norm. If w is a zero of u with $\|w\| = \|z\|$, then by (2.6), $n = \pm 1$ so $|\langle w\,, y\rangle| = \pi/2$. Hence

$$|\langle w\,, z\rangle| = \frac{\pi}{2\|y\|^2}|\langle w\,, y\rangle| = \left(\frac{\pi}{2\|y\|}\right)^2 = \|w\|\,\|z\|\,.$$

Since we have equality in Schwarz's inequality, it follows that $w = \pm z$. □

We close with some remarks concerning the proof of Theorem 1. In the first half of the proof, we showed that u was C1. Although this was not necessary for the proof of Theorem 1, it is necessary for Corollary 2.

The bulk of the proof was to show that a continuous C2 (or C1) function satisfies d'Alembert's equation. After that, the proof was fairly standard. One could proceed differently at this point by employing the following theorem [4, 12, 14].

Theorem 5. *The general complex-valued solutions of d'Alembert's functional equation on the cartesian square of an abelian group G are given by*

$$u(x) = (h(x) + h(-x))/2$$

where h satisfies $h(x+y) = h(x)h(y)$ for all $x\,,\ y \in G$.

In our case, u is continuous and it follows that h is also continuous. One can now argue that there exists a continuous linear functional $f\colon V \to \mathbb{C}$ such

that $h(x) = e^{f(x)}$ for all $x \in V$. It follows that $u(x) = \cosh f(x)$ for all $x \in V$. Letting f_1 and f_2 be the real and imaginary parts of f, respectively, we then have

$$\begin{aligned} u(x) &= \cosh[f_1(x) + i f_2(x)] \\ &= \cosh f_1(x) \cos f_2(x) - i \sinh f_1(x) \sin f_2(x). \end{aligned}$$

Since $u(x)$ is real, one can further argue that either $f_1 = 0$ or $f_2 = 0$. The amount of work required in providing the details for this method is about the same as that employed in the present proof.

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Department of Mathematics and Computer Science, University of Denver, Denver, Colorado 80208