A CLASS OF PRIMARY ABELIAN GROUPS
CHARACTERIZED BY ITS SOCLES

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ABSTRACT. The t-product of a family $\{G_i\}_{i \in I}$ of abelian p-groups is the torsion subgroup of $\prod_{i \in I} G_i$, which we denote by $\prod_{i \in I}^t G_i$. The t-product is, in the homological sense, the direct product in the category of abelian p-groups. Let $\mathscr{R}^s$ be the smallest class containing the cyclic groups that is closed with respect to direct sums, summands, and t-products. It is proven that two groups in $\mathscr{R}^s$ are isomorphic iff their socles are isomorphic as valuated vector spaces. This generalizes a classical result on direct sums of torsion-complete groups. As is frequently the case with homomorphisms defined on products, the index sets will be assumed to be nonmeasurable.

In this note, by the term "group", we will mean an abelian p-group, where p is some fixed prime. The notation and terminology will follow [F1]. We also assume some rudimentary knowledge of valuated vector spaces (e.g., [F2]).

The height function on a group G induces a valuation on the socle $G[p] = \{g \in G : pg = 0\}$. A frequently repeated theme in abelian group theory involves the relationship between the structure of G as a group and $G[p]$ as a valuated vector space. In particular, if G and H are groups whose socles are isometric (i.e., isomorphic as valuated vector spaces), when can we conclude that $G \cong H$? This question is of particular importance in the case of separable groups (i.e., those with no nonzero elements of infinite height). If G is torsion-complete or a direct sum of cyclics, then $G \cong H$ whenever $G[p]$ and $H[p]$ are isometric. However, much work has been done to show that this is almost never true if G fails to be of this type (e.g., [HM, Du, Cu, MS, S]).

To get more useful results, one has to restrict the class being considered; the class of all groups is simply too large. The question then becomes: find classes C such that whenever, G, H are in C, G is isomorphic to H iff $G[p]$ is isometric to $H[p]$. If this is the case, we will say that the groups in C are *characterized by their socles*. There are two main results of this type in the literature. In [FI] it was shown that the $p^{\omega+1}$-projectives are characterized by their socles (G is the $p^{\omega+1}$-projective iff there is a subgroup $P \subseteq G[p]$ with G/P a direct sum of cyclics). Of more relevance for this work, in [H1] it was

shown that the direct sums of torsion-complete groups are characterized by their socles. It is this result that we generalize.

The *t-product* $\{G_i\}_{i \in I}$ of a collection of groups is the torsion subgroup of $\prod G_i$, which we denote by $\prod^t G_i$. The *t*-product is clearly the direct product (in the homological sense) in the category of abelian *p*-groups. If for each positive integer n, B_n is a direct sum of copies of Z_{p^n}, then $\prod^t B_n$ is the usual way to express a torsion-complete group. In fact, in [IO] it was shown that $\prod^t G_i$ is torsion-complete iff this is true for each G_i. The *t*-product is, in certain respects, less well behaved than the usual direct product in the category of all abelian groups. For example, if A_i is a subgroup of G_i, it will frequently be the case that the induced map $\prod^t G_i \to \prod^t (G_i/A_i)$ will fail to be surjective.

Let $\mathscr{R}$ be the smallest class containing the cyclic groups, which is closed with respect to (possibly infinite) direct sums and *t*-products. This class is analogous to the *Reid class* (the smallest class containing the infinite cyclic group, closed with respect to direct sums and direct products), which has received considerable attention by torsion-free group theorists (see, for example, [DuZ, Iv]). Let $\mathscr{R}^s$ be the class of groups isomorphic to summands of groups in $\mathscr{R}$. The central result of this work (Theorem 2) states that the groups in $\mathscr{R}^s$ are characterized by their socles. Since a direct sum of torsion-complete groups is clearly in $\mathscr{R}$, this generalizes the result of [H1] mentioned above.

We will be discussing homomorphism defined on potentially large products. When this is the case, it is frequently helpful to assume that all cardinals are nonmeasurable. This use of set-theory in abelian groups goes back to [Lo]. The assumption is not overly strict, however, since measurable cardinals, if they exist at all, are extremely large. In fact, in several standard set-theoretic environments, such as the constructible universe, it can be shown that all cardinals are nonmeasurable. This assumption is necessary in that our results depend upon the following, which is a variant on a result that has appeared in various forms in [Ch, DuZ, E, Iv, K], etc.

Main Lemma. *Suppose* $\{V_i\}_{i \in I}$ *and* $\{W_j\}_{j \in J}$ *are collections of valued vector spaces with no nonzero elements of infinite value. If* $\phi \colon \prod V_i \to \bigoplus W_j$ *is a* (*valued*) *homomorphism, then there is a positive integer* n *and finite subsets* I_0, J_0 *of* I, J *such that*

$$\phi \left(\prod_{I - I_0} V_i(n) \right) \subseteq \bigoplus_{J_0} W_j.$$

Proof. The result is standard, but because of its pivotal position in this work we include an outline. If I is countable and the result failed, we could inductively construct an element of the domain that would be mapped to an element of the codomain with an infinite number of nonzero coordinates, which cannot happen. The extension to the case where I is nonmeasurable is then an exercise in Boolean algebras.

As was noted in [K], this lemma fails in the presence of measurable cardinals, so our techniques strongly depend upon the standing hypothesis of nonmeasurability.

We associate with every ordinal α subclasses $\mathscr{S}_\alpha$ and $\mathscr{T}_\alpha$ of $\mathscr{R}$ as follows: let $\mathscr{S}_0 = \mathscr{T}_0$ be the cyclic groups. Having defined $\mathscr{S}_\beta$ and $\mathscr{T}_\beta$ for all $\beta < \alpha$,

let $\mathscr{S}_\alpha$ (resp., $\mathscr{T}_\alpha$) be the groups that are isomorphic to a direct sum (resp., t-product) of a collection $\{G_i\}_{i \in I}$ where each G_i is in $\mathscr{T}_{\beta_i}$ (resp., $\mathscr{S}_{\beta_i}$) for some $\beta_i < \alpha$. Clearly $\mathscr{S}_\alpha$ and $\mathscr{T}_\alpha$ are closed with respect to direct sums and t-products, respectively. Let $\mathscr{R}_\alpha = \mathscr{S}_\alpha \cup \mathscr{T}_\alpha$. The following summarizes some elementary properties of these classes.

Lemma 1. (a) *If $\beta < \alpha$, then $\mathscr{R}_\beta \subseteq \mathscr{S}_\alpha \cap \mathscr{T}_\alpha$.*

(b) *$\mathscr{R}$ is the union of the $\mathscr{R}_\alpha$ over all ordinals α.*

(c) *If G is in any of the above classes and n is a positive integer, then $G \cong B \oplus X$, where B is a maximal p^n-bounded subgroup of G and X is in the same class as G.*

(d) *All of these classes are closed with respect to finite direct sums and finite t-products.*

Proof. (a) Clearly $\mathscr{S}_\beta \subseteq \mathscr{S}_\alpha$, and every element of $\mathscr{S}_\beta$ can be thought of as a t-product over a single element set, so $\mathscr{S}_\beta \subseteq \mathscr{T}_\alpha$, giving $\mathscr{S}_\beta \subseteq \mathscr{S}_\alpha \cap \mathscr{T}_\alpha$. Similarly, $\mathscr{T}_\beta \subseteq \mathscr{S}_\alpha \cap \mathscr{T}_\alpha$, and the results follows.

(b) We need to show that $\bigcup \mathscr{R}_\alpha$ is closed under direct sums and t-products. If $\{G_i\}_{i \in I}$ and $G_i \in \mathscr{R}_{\alpha_i}$, then by (a) each $G_i \in \mathscr{T}_{\alpha_i^+}$, so if $\alpha > \alpha_i^+$ for each i, then $\bigoplus G_i \in \mathscr{S}_\alpha$, as required. The proof for t-products is similar.

(c) If $G_i = B_i \oplus X_i$, where B_i is a maximal p^n-bounded summand of G_i, then $\bigoplus B_i$ (resp., $\prod^t B_i$) is a maximal p^n-bounded summand of $\bigoplus G_i = (\bigoplus B_i) \oplus (\bigoplus X_i)$ (resp., $\prod^t G_i = (\prod^t B_i) \oplus (\prod^t X_i)$). So in inductively building up an element of one of these classes, we can separate the terms of height less than p^n from the others, giving the result.

(d) is more or less clear, since *finite* direct sums and t-products are interchangeable.

For $G \in \mathscr{R}$, let $\mu(G)$ be the smallest α such that G is in $\mathscr{R}_\alpha$.

If G and H are groups, we say a homomorphism $\phi: G[p] \to H[p]$ is *extendable* if there is a homomorphism $\theta: G \to H$ that restricts to ϕ. If ϕ is extendable, it must be a homorphism in the category of valuated vector spaces. The following is the most important step in the proof of our main result.

Lemma 2. *If G and H are in $\mathscr{R}$ and $\phi: G[p] \to H[p]$ is a homomorphism of valuated vector spaces, then ϕ is extendable.*

Proof. We induct on $\alpha = \mu(G)$. If $\alpha = 0$, then G is cyclic, and clearly ϕ is extendable. Suppose, then, that the result is true for all groups $G' \in \mathscr{R}$ with $\mu(G') < \alpha$.

Case 1. G is in $\mathscr{S}_\alpha$. There is a collection $\{G_i\}_{i \in I}$ such that $G = \bigoplus G_i$ and for each i, $\mu(G_i) < \alpha$. If ϕ_i is the restriction of ϕ to G_i, then by induction each ϕ_i is extendable to a map θ_i, and $\theta = \sum \theta_i$ is an extension of ϕ.

Case 2. G is in $\mathscr{T}_\alpha$. In this case, $G = \prod^t G_i$, where each $G_i \in \mathscr{S}_{\alpha_i}$ for some $\alpha_i < \alpha$. We induct on $\beta = \mu(H)$. If $\beta = 0$, then H is a cyclic group, say of order p^e. Let $G = B \oplus X$, where B is a maximal p^e-bounded summand of G. Every element of $X[p]$ has height at least e, so $\phi(X[p]) = 0$. Since B is a direct sum of cyclics, it is easily verified that $\phi|_{B[p]}$ extends to $\theta: B \to H$, and letting $\theta|_X = 0$, we can conclude that ϕ is extendable. So assume the result is true for all groups $H' \in \mathscr{R}$ with $\mu(H') < \beta$.

Subcase (i). H is in $\mathscr{T}_\beta$. There is a collection $\{H_j\}_{j \in J}$ such that for each $j \in J$, $\mu(H_j) < \beta$ and $H = \prod^t H_j$. Let $\phi_j: G[p] \to H_j[p]$ be the composition

of ϕ with the projection onto $H_j[p]$. So ϕ_j is a valued homomorphism, and hence by induction it is extendable to $\theta_j\colon G \to H_j$. If $\theta\colon G \to H$ is the product of the θ_j, then we see that ϕ is also extendable.

Subcase (ii). H is in $\mathscr{S}_\beta$. In this case, $H = \bigoplus H_j$, where $H_j \in \mathscr{T}_{\beta_j}$ for some $\beta_j < \beta$. By Main Lemma, there are finite subsets I_0, J_0 of I, J, and a positive integer n such that

$$\phi\left(\prod\prod_{I-I_0}{}^t p^n G_i[p]\right) \subseteq \bigoplus_{J_0} H_j[p].$$

For each $i \in I - I_0[p]$, let $G_i = B_i \oplus X_i$, where B_i is a maximal p^n-bounded summand. Note $p^n G_i = X_i[p]$, and by Lemma 1(c), we may assume $X_i \in \mathscr{S}_{\alpha_i}$. Let

$$G_1 = \left(\prod\prod_{I_0}{}^t G_i\right) \oplus \left(\prod\prod_{I-I_0}{}^t B_i\right) \quad \text{and} \quad G_2 = \prod\prod_{I-I_0}{}^t X_i,$$

and so $G = G_1 \oplus G_2$. Note $G_2 \in \mathscr{T}_\alpha$, and by Lemma 1(d), $G_1 \in \mathscr{S}_{\alpha_0}$, where $\alpha_0 = \max_{I_0}\{\alpha_i\} < \alpha$ (the second summand is bounded and hence a direct sum of cyclics). So by induction on α, the map $\phi\colon G_1[p] \to \bigoplus_J H_j[p]$ must be extendable to a map $\theta_1\colon G_1 \to \bigoplus_J H_j$. Again by Lemma 1(d), $\bigoplus_{J_0} H_j \in \mathscr{T}_{\beta_0}$, where $\beta_0 = \max_{J_0}\{\beta_j\} < \beta$. So by induction on β, the map $\phi\colon G_2[p] \to \bigoplus_{J_0} H_j[p]$ extends to a map $\theta_2\colon G_2 \to \bigoplus_{J_0} H_j$. If θ is the sum of θ_1 and the composition of θ_2 with the usual inclusion, then θ will extend ϕ, as desired.

Our immediate aim is to extend this lemma to $\mathscr{R}^s$. We pause for the following characterization of this class.

Proposition 1. *$\mathscr{R}^s$ is the smallest class containing the cyclic groups that is closed with respect to the formation of direct sums, summands, and t-products.*

Proof. Since $\mathscr{R}^s$ is trivally closed with respect to summands, we need to show that it is also closed with respect to direct sums and t-products. But if $\{G_i\}_{i\in I}$ is a family of groups in $\mathscr{R}^s$, then each G_i is the summand of a group X_i in $\mathscr{R}$ and so $\bigoplus G_i$ and $\prod{}^t G_i$ are summands of $\bigoplus X_i$ and $\prod{}^t X_i$, respectively, and these latter groups are in $\mathscr{R}$.

Lemma 3. *If G and H are in $\mathscr{R}^s$, then there is an $X \in \mathscr{R}$ such that $G \oplus X$ and $H \oplus X$ are in $\mathscr{R}$.*

Proof. Suppose $G \oplus Y$ and $H \oplus Z$ are in $\mathscr{R}$. Consider

$$X = \left(\bigoplus_\omega G\right) \oplus \left(\bigoplus_\omega Y\right) \oplus \left(\bigoplus_\omega H\right) \oplus \left(\bigoplus_\omega Z\right)$$

$$\cong \left(\bigoplus_\omega (G \oplus Y)\right) \oplus \left(\bigoplus_\omega (H \oplus Z)\right).$$

So $G \oplus X \cong H \oplus X \cong X$ is in $\mathscr{R}$.

Theorem 1. *If G and H are in $\mathscr{R}^s$ and $\phi\colon G[p] \to H[p]$ is a valuated vector space homomorphism, then ϕ is extendable.*

Proof. Using Lemma 3, choose $X \in \mathscr{R}$ so that $G \oplus X$ and $H \oplus X$ are in $\mathscr{R}$. Define $\gamma\colon (G \oplus X)[p] \to (H \oplus X)[p]$ to be ϕ on the first summands and

the identity map on the second. By Lemma 2, γ extends to a homomorphism, $\lambda: G \oplus X \to H \oplus X$. Letting $\theta: G \to H$ be the composition

$$G \to G \oplus X \xrightarrow{\lambda} H \oplus X \to H,$$

clearly θ extend ϕ, as desired.

Lemma 4. *If $\phi: G[p] \to H[p]$ extends to $\theta: G \to H$, then θ is an isomorphism iff ϕ is an isometry.*

Proof. Sufficiency being obvious, suppose ϕ is an isomorphism of valuated vector spaces. If the kernel of θ was nonzero, it would have an element of order p. This element would also have to be in the kernel of ϕ, which is a contradiction. So θ is injective. Consider $G' = \theta(G)$. Since ϕ preserves values, it follows that G' is a pure subgroup of H (see [F1, 26(h)]). But since $H[p] = \phi(G[p]) = G'[p]$, it follows that $G' = H$ (see [F1, 26(j)]), so θ is also surjective.

Theorem 2. *The groups in $\mathscr{R}^s$ are characterized by their socles. In fact, any isometry between the socles of groups in $\mathscr{R}^s$ extends to an isomorphism of the groups.*

Proof. Theorem 1 and Lemma 4.

Let $\mathscr{S}_\alpha^s$ and $\mathscr{T}_\alpha^s$ be the classes of groups isomorphic to summands of $\mathscr{S}_\alpha$ and $\mathscr{T}_\alpha$, respectively. We now wish to show that for any ordinal α, there is a $G \in \mathscr{R}$ with $\mu(G) = \alpha$. In doing so, we will actually construct two sequences of groups of increasing complexity. Note that $\mathscr{S}_1 = \mathscr{S}_1^s$ is the direct sums of cyclics and $\mathscr{T}_1 = \mathscr{T}_1^s$ is the torsion-completes. So $\mathscr{S}_1 \cap \mathscr{T}_1$ is the bounded groups. For higher ordinals, we can construct examples using the following (cf., [DuZ, Iv]):

Theorem 3. *Suppose $1 < \alpha$ is an ordinal and for every $0 < i < \alpha$, we have groups $G_i \in \mathscr{T}_i - \mathscr{S}_i^s$ and $H_i \in \mathscr{S}_i - \mathscr{T}_i^s$. Then*
  (a) $H_\alpha = \bigoplus_\omega \bigoplus_{i<\alpha} G_i$ *is in $\mathscr{S}_\alpha - \mathscr{T}_\alpha^s$,*
  (b) $G_\alpha = \prod_\omega^t \prod_{i<\alpha}^t H_i$ *is in $\mathscr{T}_\alpha - \mathscr{S}_\alpha^s$.*

Proof. Clearly $H_\alpha \in \mathscr{S}_\alpha$ and $G_\alpha \in \mathscr{T}_\alpha$.

Considering (a), suppose H_α is a summand of $\prod_J^t Y_j$, where each $Y_j \in \mathscr{S}_{\alpha_j}$ for some $\alpha_j < \alpha$, π is the projection onto H_α, and $Z = \ker \pi$ is the complementary summand. By Main Lemma, there are finite sets K, J_0 of ω, J and a positive integer n such that

$$\pi \left(\prod_{J-J_0} \prod^t p^n Y_j[p] \right) \subseteq \bigoplus_K \bigoplus_\alpha G_i[p].$$

Letting $M_0 = \bigoplus_K \bigoplus_\alpha G_i$ and $M_1 = \bigoplus_{\omega-K} \bigoplus_\alpha G_i$, we have

$$(*) \qquad \prod_{J-J_0} \prod^t p^n Y_j[p] \subseteq Z[p] \oplus M_0[p].$$

For each $j \in J$, let B_j be a maximal p^n-bounded summand of Y_j and $Y = \prod_{J-J_0}^t B_j \oplus \prod_{J_0}^t Y_j$. Since the first term is bounded (and hence a direct sum of cyclics) and the second product is finite, there is a $\beta < \alpha$ such that $Y \in \mathscr{S}_\beta$.

Note that $(*)$ implies that $M_1[p]$ is isometric to a summand of $Y[p]$, so M_1 is isomorphic to a summand of Y by Theorem 2. But M_1 contains a summand isomorphic to G_β, which by assumption is not isomorphic to a summand of a group in $\mathscr{S}_\beta$. The resulting contradiction proves (a).

Considering (b), suppose G_α is a summand of $\bigoplus_J E_j$, where $E_j \in \mathscr{T}_{\alpha_j}$ for some $\alpha_j < \alpha$. By applying Main Lemma to the inclusion map, there are finite subsets K, J_0 of ω, J and a positive integer n such that

$$\prod_{\omega-K} {\prod_{\alpha}}^t \prod \prod^t p^n H_i[p] \subseteq \bigoplus_{J_0} E_j[p].$$

Since J_0 is finite, there is a $\beta < \alpha$ such that $E = \bigoplus_{J_0} E_j \in \mathscr{T}_\beta$. If $H_\beta = B \oplus X$ with B a maximal p^n-high subgroup, then $X[p] = p^n H_\beta[p]$ will be a (valuated) summand of $E[p]$, and hence by Theorem 2, $H_\beta = B \oplus X$ will be isomorphic to a summand of $B \oplus E \in \mathscr{T}_\beta$, contrary to assumption.

Corollary 1. *For every ordinal α there is a G with $\mu(G) = \alpha$.*

The groups G_α and H_α constructed above are unnecessarily large. They were chosen mostly to simplify notation. The interested reader can check that when α is a limit ordinal, we could let

$$G_\alpha = \bigoplus_{i<\alpha} H_i \quad \text{and} \quad H_\alpha = \prod \prod_{i<\alpha}^t G_i,$$

and when α is isolated, we could let

$$G_\alpha = \bigoplus_{\omega} H_{\alpha-1} \quad \text{and} \quad H_\alpha = \prod \prod_{\omega}^t G_{\alpha-1},$$

and the above proof goes through without significant change.

Corollary 2. *The groups G_α, H_α constructed above are all nonisomorphic.*
Proof. They clearly belong to different classes.

We conclude with a couple of questions.
(1) To what extent do the above results hold in the presence of measurable cardinals? Recall that Main Lemma is definitely not valid in this case.
(2) For an ordinal α, does $\mathscr{S}_\alpha = \mathscr{S}_\alpha^s$? Similarly, does $\mathscr{T}_\alpha = \mathscr{T}_\alpha^s$? This is clear when $\alpha = 1$. In [H2] it was shown that a summand of a group isomorphic to a direct sum of torsion-completes is also of this form. In other words, $\mathscr{S}_2 = \mathscr{S}_s^2$. In [La] It was shown that a summand of a *countable* t-product of groups that are direct sums of cyclics will once again be of this form. This is evidence that $\mathscr{T}_2 = \mathscr{T}_2^s$ is valid, but the proof utilized in [La] strongly depends upon the countability of the t-product.

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