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GROUPS WITH MANY REWRITABLE PRODUCTS

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ABSTRACT. For any integer $n \geq 2$, denote by R_n the class of groups G in which every infinite subset X contains n elements $x_1, \ldots, x_n$ such that the product $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$ for some permutation $\sigma \neq 1$. The case $n = 2$ was studied by B. H. Neumann who proved that R_2 is precisely the class of centre-by-finite groups. Here we show that $G \in R_n$ for some n if and only if G contains an FC-subgroup F of finite index such that the exponent of $F/Z(F)$ is finite, where $Z(F)$ denotes the centre of F.

INTRODUCTION

B. H. Neumann proved in [8] that a group is centre-by-finite if and only if every infinite subset contains a pair of elements that commute. Extensions of problems of this type are to be found in [7] and [6]. The notion of commutativity was extended to rewritable products in [4] with a complete description obtained in [5]. Detailed study of rewritable groups may be found in papers by R. Blyth [1] and [2]. Following earlier authors, we call G a P_n-group if given any sequence $x_1, \ldots, x_n$ of n elements in G, their product $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$ for some permutation $\sigma \neq 1$. A group G is called a Q_n group if given any set $\{x_1, \ldots, x_n\}$ of n elements of G, $x_{\sigma(1)} \cdots x_{\sigma(n)} = x_{\phi(1)} \cdots x_{\phi(n)}$ for some permutations $\sigma \neq \phi$. Clearly $P_n \subseteq Q_n$ and Blyth [1] has shown that $\bigcup_n P_n = \bigcup_n Q_n$.

Call G an R_n group if every infinite subset X of G contains a subset $\{x_1, \ldots, x_n\}$ of n elements such that $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$ for some permutation $\sigma \neq 1$. Clearly $Q_n \subseteq R_n$ for every n. The relation of R_n to Q_n is rather like that of centre-by-finite groups to abelian groups. Our main result is the following.

Theorem A. *A group G is an R_n group for some integer n if and only if G has a normal subgroup F such that G/F is finite, F is an FC-group and the exponent of $F/Z(F)$ is finite.*

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Recall that a group G is an FC-group if every element of G has only finitely many conjugates in G. For results on FC-groups we refer the reader to [9].

PROOFS

Groups G in which, given any infinite sequence $x_1, x_2, \ldots$ of elements in G, there exist some $m \in \mathbb{N}$ and permutations $\sigma \neq \phi$ in S_m such that

$$x_{\sigma(1)} \cdots x_{\sigma(m)} = x_{\phi(1)} \cdots x_{\phi(m)}$$

were studied in [3]. The class Q_∞ of all such groups was shown to be precisely the class of FC-by-finite groups. Our first lemma makes use of this.

Lemma 1. *If $G \in R_n$ for some n then G is FC-by-finite.*

Proof. Let $G \in R_n$ and let $x_1, x_2, \ldots$ be any infinite sequence. Let $X = \{x_i;\ i = 1, 2, \ldots\}$. It contains n elements, say $x_{\lambda_1}, \ldots, x_{\lambda_n}$ such that their product is rewritable. Let $\lambda = \max\{\lambda_i;\ i = 1, \ldots, n\}$. Consider the sequence $x_1, \ldots, x_\lambda$. Reorder the sequence as $y_1, \ldots, y_\lambda$ where $y_i = x_{\lambda_i}$, $i = 1, \ldots, n$ and the rest of the y_i's are the rest of x_i's in some order.

Now $y_1 \cdots y_n = y_{\sigma(1)} \cdots y_{\sigma(n)}$ for some $\sigma \neq 1$. Hence $y_1 \cdots y_n y_{n+1} \cdots y_\lambda = y_{\sigma(1)} \cdots y_{\sigma(n)} y_{n+1} \cdots y_\lambda$ and so $G \in Q_\infty$. Thus by Proposition 2 of [3], G is FC-by-finite.

Lemma 2. *Let F be a normal subgroup of a group G such that G/F is finite of exponent e, $G/Z(F)$ is of finite exponent e' and F is an FC-group. Then $G \in R_{e+e'}$.*

Proof. Since every finite group is in R_n for every $n \geq 2$, we may assume that G is not finite. Let X be any infinite set in G. Then $|Fg \cap X| = \infty$ for some $g \in G$. Thus we may as well assume that $X \subseteq Fg$. Pick any $x_1, \ldots, x_e$ from X. Their product $x_1 \cdots x_e = f$ lies in F. Let $X' = X \backslash \{x_1, \ldots, x_e\}$. Since F is an FC-group and $f \in F$, the centralizer C of $\langle f^G \rangle$ in F is of finite index in G; also $C \supseteq Z(F)$. Now $|X' \cap Ch| = \infty$ for some h in G so pick any e' elements $c_1 h, \ldots, c_{e'} h$ from $X' \cap Ch$. Then

$$f(c_1 h) \cdots (c_{e'} h) = (c_1 h) \cdots (c_{e'} h) f^{h^{e'}} = (c_1 h) \cdots (c_{e'} h) f$$

for $h^{e'} \in Z(F)$. Since f is a product of e elements of X, $G \in R_{e+e'}$.

The above lemma provides the easy half of the proof of Theorem A. For the other half we need the following preliminary results.

Lemma 3. *Let G be an FC group that is also nilpotent of class two. Then $G \in R_n$ for some $n \geq 2$ implies $G/Z(G)$ has finite exponent.*

Proof. Suppose not. Then there exist a_1, b_1 in G such that $[b_1^i, a_1] = c_1^i \neq 1$ for all $i = 1, \ldots, n$. Clearly if we set $\bar{G} = G/\langle c_1 \rangle$, the exponent of the group $\bar{G}/Z(\bar{G})$ is not finite, for G is an FC-group so that $\langle c_1 \rangle$ is finite. Next choose b_2, a_2 in $C_G\langle a_1, b_1 \rangle$ such that $[b_2^i, a_2] = c_2^i \notin \langle c_1 \rangle$ for all $i = 1, \ldots, n$. This is possible since $C_G\langle a_1, b_1 \rangle$ is of finite index in G.

Continue; at step k, pick b_k, a_k in $C_G\langle a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1} \rangle$ such that $[b_k^i, a_k] = c_k^i$ does not lie in $\langle c_1, \ldots, c_{k-1} \rangle$ for all $1 \leq i \leq n$. Now put $x_0 = b_1$ and $x_k = a_1 \ldots a_k b_{k+1}$ for $k > 0$. Take $X = \{x_i;\ i \geq 0\}$. We claim that if $\lambda_1, \ldots, \lambda_n$ are any n distinct nonnegative integers, then $x_{\lambda_1} \cdots x_{\lambda_n} = x_{\lambda_{\sigma(1)}} \cdots x_{\lambda_{\sigma(n)}}$ implies $\sigma = 1$.

For each λ_j let $r(\lambda_j)$ denote the number of x_{λ_i}'s with $\lambda_i > \lambda_j$ appearing to the right of x_{λ_j} in the expression $x_{\lambda_1} \cdots x_{\lambda_n}$. Let $s(\lambda_j)$ denote the number of x_{λ_i}'s with $\lambda_i > \lambda_j$ appearing to the right of x_{λ_j} in the expression $x_{\lambda_{\sigma(1)}} \cdots x_{\lambda_{\sigma(n)}}$. Let λ_j be the largest integer such that $r = r(\lambda_j) \neq s(\lambda_j) = s$. Then modulo $\langle c_1, \ldots, c_{\lambda_j} \rangle$,

$$(x_{\lambda_1} \ldots x_{\lambda_n})(x_{\lambda_{\sigma(1)}} \ldots x_{\lambda_{\sigma(n)}})^{-1} \equiv b_k a_k^r a_k^{-s} b_k^{-1} a_k^{s-r} \equiv c_k^{r-s},$$

where $k = \lambda_j + 1$.

Since $1 \leq |s-r| < n$, $c_k^{s-r} \notin \langle c_1, \ldots, c_{\lambda_j} \rangle$ and hence $x_{\lambda_1} \cdots x_{\lambda_n} = x_{\lambda_{\sigma(1)}} \cdots x_{\lambda_{\sigma(n)}}$ only if $r(\lambda_j) = s(\lambda_j)$ for all j. But this implies that $\sigma = 1$ and thus completes the proof.

Lemma 4. *Let G be an FC-group such that the exponent of $G/\zeta_2(G)$ is finite. If $G \in R_n$ for some $n \geq 2$, then the exponent of $G/\zeta_1(G)$ is finite. ($\zeta_i(G)$ denotes the ith centre of G).*

Proof. Write Z_i for $\zeta_i(G)$. By Lemma 3, the exponent of $Z_2/\zeta_1(Z_2)$ is finite. Put $A = \zeta_1(Z_2)$.

It will suffice to show that the exponent of A/Z_1 is finite. Let e be the exponent of G/Z_2 and suppose there is $a \in A$ such that $a^e \notin Z_1$. Then $[a^e, b] \neq 1$ for some $b \in G$. But $[a^e, b] = [a, b]^e = [a, b^e] \in [A, Z_2] = 1$, a contradiction.

Proof of Theorem A. Let $G \in R_n$ for some $n \geq 2$. By Lemma 1, the FC-centre F of G is of finite index in G. Because of Lemma 2, we only need to show that $F/Z(F)$ has finite exponent to complete the proof. If F is finite, there is nothing to prove, so we may assume $|F| = \infty$. If $F/\zeta_2(F)$ has finite exponent, then Lemma 4 applies and we are done.

If $F/\zeta_2(F)$ is not of finite exponent then consider the group $\bar{F} = F/Z(F)$ and get a contradiction by showing that the exponent of $\bar{F}/Z(\bar{F})$ is finite. The purpose of considering $F/Z(F)$ is that it is residually finite (see Theorem 1.9 of [9]). So we assume that F is a residually finite FC-group in the class R_n for some $n \geq 2$ and that the exponent of $F/Z(F)$ is not finite. Then there exist a_1, b_1 in F such that $[a_1^i, b_1] \neq 1$ for all $i = 1, 2, \ldots, n$. Let $A_1 = \langle [a_1, b_1]^F \rangle$. Note that A_1 is finite. Let $C_1 = C_F(\langle a_1, b_1 \rangle^F)$ so that C_1 is of finite index in F and there exists $N_1 \lhd G$ such that $N_1 \leq C_1$, $N_1 \cap A_1 = 1$ and F/N_1 is finite. Note that $N_1/Z(N_1)$ is not of finite exponent for $F = \langle N_1, Y_1 \rangle$ for some finite set Y_1 and $C_F(Y_1)$ has finite index so that $C_F(Y_1) \cap Z(N_1) \leq Z(F)$ is of finite index in $Z(N_1)$ and hence $F/Z(F)$ would have finite exponent.

Next choose a_2, b_2 in N_1 such that $[a_2^i, b_2] \neq 1$ for all $0 < i \leq n$. Obtain subgroup $N_2 \lhd F$ such that $N_2 \leq N_1 \cap C_F(\langle a_2, b_2 \rangle^F)$, $N_2 \cap \langle [a_2, b_2]^F \rangle = 1$, and F/N_2 is finite. This is possible and N_2 is obtained in a similar way to N_1. Also note that the exponent of $N_2/Z(N_2)$ is not finite.

Continue the above process to get the sequence $a_3, b_3, a_4, b_4 \cdots$. Now put $x_0 = b_1$, $x_1 = a_1 b_2, \ldots, x_k = a_1 \cdots a_k b_{k+1}, \ldots$ and let $X = \{x_i, i \geq 0\}$. We claim that if $\lambda_1, \ldots, \lambda_n$ are any n distinct nonnegative integers and

$$x_{\lambda_1} \cdots x_{\lambda_n} = x_{\lambda_{\sigma(1)}} \cdots x_{\lambda_{\sigma(n)}}$$

then $\sigma = 1$. This is done by using an argument similar to that in Lemma 3. This completes the proof.

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